On bases of tropical Plücker functions[1]

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Abstract. We consider functions $f : B \to \mathbb{R}$ that obey tropical analogs of classical Plücker relations on minors of a matrix. The most general set B that we deal with in this paper is of the form $\{x \in \mathbb{Z}^n : 0 \le x \le a,\ m \le x_1 + \ldots + x_n \le m'\}$ (a rectangular integer box "truncated from below and above"). We construct a basis for the set $\mathcal{T}$ of tropical Plücker functions on B, a subset $\mathcal{B} \subseteq B$ such that the restriction map $\mathcal{T} \to \mathbb{R}^{\mathcal{B}}$ is bijective. Also we characterize, in terms of the restriction to the basis, the classes of submodular, so-called skew-submodular, and discrete concave functions in $\mathcal{T}$, discuss a tropical analogue of the Laurentness property, and present other results.

Keywords: Plücker relation, tropicalization, octahedron recurrence, submodular function, rhombic tiling, Laurent phenomenon

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1 Introduction

There are nice algebraic relations on minors of a matrix that have been established long ago. For a positive integer n, let $[n]$ denote the ordered set of n elements $1, 2, \ldots, n$. For an $n \times n$ matrix M and a set $J \subseteq [n]$, let Δ_J denote the determinant of the submatrix of M formed by the column set J and the row set $\{1, \ldots, |J|\}$. Then: (i) for any triple $i < j < k$ in $[n]$ and $X \subseteq [n] - \{i, j, k\}$,

$$\Delta_{Xik}\Delta_{Xj} = \Delta_{Xij}\Delta_{Xk} + \Delta_{Xi}\Delta_{Xjk};$$

and (ii) for any quadruple $i < j < k < \ell$ in $[n]$ and $X \subseteq [n] - \{i, j, k, \ell\}$,

$$\Delta_{Xik}\Delta_{Xj\ell} = \Delta_{Xij}\Delta_{Xk\ell} + \Delta_{Xi\ell}\Delta_{Xjk},$$

where for brevity we write $Xi' \ldots j'$ instead of $X \cup \{i'\} \cup \ldots \cup \{j'\}$. These equalities represent simplest cases of so-called *Plücker's relations*. (About classical Plücker's type quadratic relations involving flag minors of a matrix, see, e.g., [7]).

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Relations as above can be stated in an abstract form; namely, one can consider a function g on the *Boolean (hyper)cube* $\{0,1\}^{[n]}$ or on an appropriate part of it and impose the condition

$$g(Xik)g(Xj) = g(Xij)g(Xk) + g(Xi)g(Xjk),$$

and/or

$$g(Xik)g(Xjl) = g(Xij)g(Xk\ell) + g(Xi\ell)g(Xjk),$$

for X, i, j, k, ℓ as above (identifying a subset of $[n]$ with the corresponding 0,1 vector). Such a g is said to be an *algebraic Plücker function*, or an *AP-function*. For simplicity, in what follows we will deal with only real-valued functions.

Tropical analogs of these relations, coming up when multiplication is replaced by addition and addition is replaced by taking maximum, are viewed as

$$f(Xik) + f(Xj) = \max\{f(Xij) + f(Xk),\ f(Xi) + f(Xjk)\}, \tag{1}$$

and

$$f(Xik) + f(Xjl) = \max\{f(Xij) + f(Xk\ell),\ f(Xi\ell) + f(Xjk)\}, \tag{2}$$

(see, e.g., [1, Sec. 2]), and a function f obeying (1) and (2) is said to be a *tropical Plücker function*.

In this paper we do not restrict ourselves by merely the Boolean case, but consider functions defined on a set of a more general form, namely, on a box or a truncated box in $\mathbb{Z}^{[n]}$, and satisfying natural generalizations of (1) and (2) to such domains.

More precisely, let $a = (a_1, ..., a_n)$ be an n-tuple of natural numbers; we refer to $|a| := a_1 + \ldots + a_n$ as the *size* of a. The *a-box* is the set $B(a)$ of integer vectors $x = (x_1, ..., x_n)$ satisfying the box constraints $0 \le x_i \le a_i$ for all $i \in [n]$. In particular, the Boolean cube $2^{[n]}$ is just the box $B(\mathbf{1})$, where $\mathbf{1}$ is the all-unit vector. Given integers $0 \le m \le m' \le |a|$, by the *truncated box* bounded by m, m' we mean the subset of vectors $x \in B(a)$ with $m \le |x| \le m'$. It is denoted by $B_m^{m'}(a)$ and the number $m' - m$ is regarded as its *width*. So $B(a) = B_0^{|a|}(a)$. For $m \le p \le m'$, the pth *layer* of $B_m^{m'}(a)$ is formed by the vectors of size p in it. Sometimes we will deal with a shifted box $B(a' | a'') := \{x \in \mathbb{Z}^n \colon a' \le x \le a''\}$, where $a', a'' \in \mathbb{Z}^n$ and $a' \le a''$.

Three special cases will be important to us. When a is all-unit, we obtain the *truncated Boolean cube* $B_m^{m'}(\mathbf{1})$. When $m = m'$, we obtain a truncated box with zero width; it may be denoted as $B_m(a)$ and called a *slice*. When, in addition, $a = \mathbf{1}$, the slice turns into the *hyper-simplex* $\{S \subseteq [n] \colon |S| = m\}$.

Definition. Let B be a truncated box $B_m^{m'}(a)$. A function $f : B \to \mathbb{R}$ is called a *tropical Plücker function*, or a *TP-function* for short, if it satisfies the *TP3-relation*:

$$\begin{aligned} &f(x + 1_i + 1_k) + f(x + 1_j) \\ &\qquad = \max\{f(x + 1_i + 1_j) + f(x + 1_k),\ f(x + 1_i) + f(x + 1_j + 1_k)\} \end{aligned} \tag{3}$$

for any x and $1 \le i < j < k \le n$, and satisfies the *TP4-relation*:

$$\begin{aligned} &f(x + 1_i + 1_k) + f(x + 1_j + 1_\ell) \\ &= \max\{f(x + 1_i + 1_j) + f(x + 1_k + 1_\ell),\ f(x + 1_i + 1_\ell) + f(x + 1_j + 1_k)\} \end{aligned} \tag{4}$$

for any x and $1 \le i < j < k < \ell \le n$, provided that all six vectors occurring as arguments in (3) belong to B, and similarly for (4). Here 1_q denotes qth unit base vector in $\mathbb{R}^{[n]}$.

So each TP4-relation concerns vectors of the same layer and vanishes in pth layer for $p = 0, 1, |a|-1, |a|$ or when $n < 4$, while each TP3-one concerns vectors of two neighboring layers. The zero vector (as well as a) occurs in no relation at all and we will often assume that $f(\mathbf{0}) = 0$. When $a = \mathbf{1}$, (3) and (4) turn into (1) and (2), respectively. Sometimes, when all six vectors occurring as arguments in (3) belong to B, we will refer to (x, i, j, k) as a *(feasible) cortege*, and similarly for (x, i, j, k, ℓ) (concerning (4)).

Functions satisfying algebraic or tropical Plücker-type relations have been studied in literature. Such functions on Boolean cubes are considered by Berenstein, Fomin and Zelevinsky [1] in connection with the total positivity and Lusztig's canonical bases; see also [12]. Henriques [10] considers AP-functions on the set of integer solutions of the system $0 \le x_i \le m-1$, $x_1 + \ldots + x_n = m$, and refers to the work of Fock and Goncharov [6] for results on such functions. The tropical analogs of those AP-functions, with one additional condition (of rhombic concavity) imposed on them, form a class of polymatroidal concave functions, or M-functions, studied by Murota [15]; see also [14]. Tropical Plücker functions in dimension 3 and 4 are considered in [3, 13, 19] in connection with the so-called octahedron recurrence. An instance of Plücker relations is a relation on six lengths between four horocycles in the hyperbolic plane with distinct centers at infinity [16]. The TP4-functions on a hyper-simplex form a special case of so-called *valuated matroids* introduced by Dress and Wenzel [4].

The set of TP-functions on $B = B_m^{m'}(a)$ is denoted by $\mathcal{T}(B)$. In this paper we give an explicit construction of a basis for the TP-functions on B.

Definition. A subset $\mathcal{B} \subseteq B = B_m^{m'}(a)$ is called a *TP-basis*, or simply a *basis*, if the restriction map $res : \mathcal{T}(B) \to \mathbb{R}^{\mathcal{B}}$ is a bijection. In other words, each TP-function on B is determined by by values on $\mathcal{B}$, and moreover, values on $\mathcal{B}$ can be chosen arbitrarily.

Showing that such a basis does exist, we construct a basis of a rather simple form, as follows.

For a nonzero vector $x \in B(a)$, let $c(x)$ and $d(x)$ denote, respectively, the first and last elements (w.r.t. the order in $[n]$) in the support $\mathrm{supp}(x) = \{i : x_i \ne 0\}$ of x. We say that x is a *fuzzy-interval*, or, briefly, a *fint*, if $x_i = a_i$ for all $c(x) < i < d(x)$. We say that x is a *sesquialteral fuzzy-interval*, or a *sint*, if x is not a fint and is representable as the sum of two fints x', x'' such that $d(x') < c(x'')$, and $x'_i = a_i$ for $i = 1, \ldots, d(x') - 1$. When $a = \mathbf{1}$, a fint turns into an *interval* $\{c, c+1, \ldots, d\}$ in $[n]$, denoted as $[c..d]$, and a sint turns into a *sesquialteral interval*, a set of the form $[1..d_1] \cup [c_2..d_2]$ with $c_2 > d_1 + 1$.

Let $Int(a; p)$ and $Sint(a; p)$ denote the sets of fints and sints x of size $|x| = p$ in $B(a)$, respectively. Our main theorem is the following.

Theorem A *The set* $\mathcal{B} := Sint(a; m) \cup Int(a; m) \cup Int(a; m+1) \cup \ldots \cup I(a; m')$ *is*

a TP-basis for the truncated box $B_m^{m'}(a)$.

We call the basis $\mathcal{B}$ figured in this theorem *standard*. (When $m = 0$, we default include the zero vector in the basis as well, without indicating it explicitly.) Observe that the standard basis involves sints only from the lowest layer. In particular, the set $Int(a;1) \cup \ldots \cup Int(a;n)$ gives a basis for the box $B(a)$, and $Sint(a;m) \cup Int(a;m)$ gives a basis for the slice $B_m(a)$. Due to this theorem, $\mathcal{T}(B)$ with $B = B_m^{m'}(a)$ is representable as a $|\mathcal{B}|$-dimensional polyhedral conic complex (a fan) in $\mathbb{R}^B$. It contains a large lineal since any quasi-separable function of the form $\varphi_1(x_1) + ... + \varphi_n(x_n) + \varphi_0(x_1 + ... + x_n)$ (where φ_i is an arbitrary function in one variable) is a TP-function of which addition to any other TP-function maintains the TP-relations.

To illustrate the theorem, consider the hyper-simplex H for $n = 4$ and $m = 2$; it consists of six sets, which may be denoted as 12, 13, 14, 23, 24, 34. By adding an appropriate quasi-separable function, any TP-function f on H is transformed so as to take zero value on the points 12, 13, 14, 24. Then the unique TP4-relation (concerning 13, 24) implies $\max\{f(23), f(34)\} = 0$. This means that, modulo the lineal, the set of TP-function is represented as the union of two rays in $\mathbb{R}^2$, namely, $(\mathbb{R}_-, 0)$ and $(0, \mathbb{R}_-)$, and therefore, it is piecewise-linear-morphic to the line $\mathbb{R}$.

An easy consequence of Theorem A is that a TP-function f on a truncated box $B_m^{m'}(a)$ can be extended to a TP-function on the entire box $B(a)$. Indeed, first we take the restriction of f to the standard basis for $B_m^{m'}(a)$ and extend it to the standard basis for $B_m^{|a|}(a)$ by assigning arbitrary values on $Int(a; m'+1) \cup \ldots \cup Int(a; |a|)$. This determines a TP-function g on $B_m^{|a|}(a)$ coinciding with f on $B_m^{m'}(a)$. Then we consider the *complementary* function g^* for g, which is defined on the vectors $x \in B_0^{|a|-m}(a)$ by $g^*(x) := g(a - x)$; clearly g^* is a TP-function again. Finally, we extend g^* into a TP-function h on $B_0^{|a|}(a) = B(a)$ by acting as at the first step. Then h^* is the desired extension of f to $B(a)$.

Special cases of Theorem A have appeared in some earlier works. The corresponding result for Boolean cubes is given in [1], and for hyper-simplexes in [17]; see also [18, 20]. The algebraic version in the case of a "simplicial" slice $\{x \in \mathbb{Z}_+^n \colon \sum x_i = m\}$ was announced in [10] with a claim that it could be obtained by use of results on cluster algebras in [6].

Our proof of Theorem A is direct and relatively short. It consists of three stages. At the first stage, we prove that the corresponding restriction map res is injective. At the second stage, we prove the surjectivity of res for the Boolean version (Theorem A′ in Section 3). At the third stage, we reduce the general case to the Boolean one. The core of the whole proof is a *flow-in-matrix* method, which consists in representing *any* TP-function f on a truncated Boolean cube by use of maximum weight flows (systems of paths) in a weighted grid associated with an $|a| \times m'$ matrix whose entries are determined by the values of f on the intervals and the m-sized sesquialteral intervals in $[n]$.

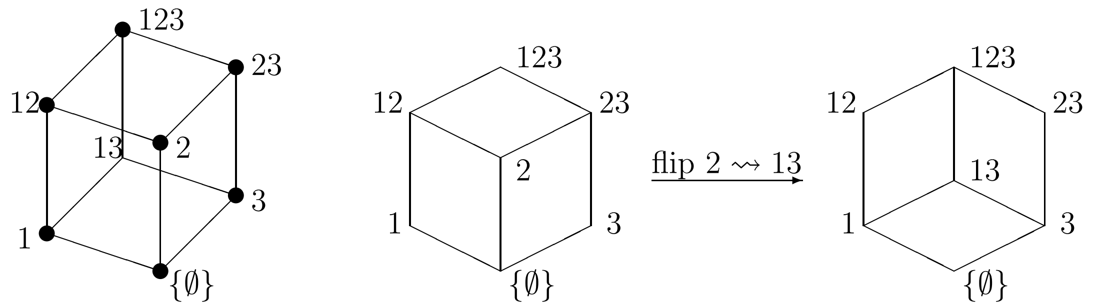
Another group of results in this paper concerns an interrelation between TP-bases and rhombic tilings, and characterizations of special classes of TP-functions.

Given a basis $\mathcal{B}$ (e.g., the standard one), we can produce more bases by making

a series of elementary transformations relying on TP3- or TP4-relations, referring to them as *mutations*, or *flips*. More precisely, suppose there is a cortege (x, i, j, k) such that the four vectors occurring in the right hand side of (3) and one vector y in the left hand side, say, $y = x + 1_j$, belong to $\mathcal{B}$. It is easy to see that the replacement in $\mathcal{B}$ of y by the other vector in the left hand side, namely, $x + 1_i + 1_k$, results in a basis as well; we can further transform the latter basis in a similar way. Analogous transformations via TP4-relations can be applied to corresponding corteges (x, i, j, k, ℓ).

When dealing with an entire box $B(a)$ (in particular, with the cube $2^{[n]}$), the standard basis, as well as many other (but not all) bases obtained from it by a series of TP3-mutations, can be associated with a *rhombic tiling diagram* on a $2n$-gone (a zonogon), giving a nice visualization of the basis. (For rhombic tilings, see, e.g., [5].)

To illustrate this, consider the cube $2^{[3]}$. The standard basis $\mathcal{B}$ for it consists of the six intervals $1, 2, 3, 12, 23, 123$ to which we also add the empty interval $\{\emptyset\}$. There is only one basis $\mathcal{B}'$ different from $\mathcal{B}$; it is obtained from $\mathcal{B}$ by the TP3-mutation $2 \rightsquigarrow 13$. The cube and the rhombic tiling diagrams for $\mathcal{B}$ and $\mathcal{B}'$ are drawn in the picture:



We refer to a TP-basis for $B(a)$ that corresponds to a rhombic tiling as a *normal* one. As an additional result, we give necessary and sufficient conditions on a subset of $B(a)$ that can be extended into a normal basis, and develop a polynomial-time algorithm to find such a basis (a similar problem for the class of all bases seems to be more sophisticated).

Using the correspondence between the normal bases and rhombic tilings, we then study the classes of *submodular* and *skew-submodular* TP-functions f on a box $B(a)$, which means that f satisfies the inequalities of the form

$$f(x + 1_i) + f(x + 1_j) \geq f(x) + f(x + 1_i + 1_j)$$

in the former case, and of the form

$$f(x + 1_i + 1_j) + f(x + 1_j) \geq f(x + 1_i) + f(x + 2 \cdot 1_j)$$

in the latter case. It turns out that each class admits a characterization in terms of the restriction of f to the standard basis (or even to an arbitrary normal basis) $\mathcal{B}$. More precisely, we show that for a TP-function f, the above submodular (skew-submodular) inequalities are propagated by the TP3-recurrence, starting from such

inequalities within $\mathcal{B}$. Furthermore, we explain that when both the submodular and skew-submodular inequalities hold, the function f possesses the property of *discrete concavity* (more precisely, polymatroidal concavity, in the sense that for the minimal concave function g on the convex hull of $B(a)$ such that $g_{|B(a)} \ge f$, all affine regions of g are generalized polymatroids).

Finally, returning to the flow-in-matrix method and considering the TP-functions f on the cube $2^{[n]}$, we show that for each set $X \subset [n]$ not in the standard basis, the value $f(X)$ is expressed as a piece-wise linear convex function h (invariant of f) of which arguments are the values on the standard basis. A similar property is shown to take place in case of truncated cubes and entire boxes. This behavior of TP-functions with respect to the standard basis can be regarded as exhibiting a tropical analogue of the so-called *Laurent phenomenon* (for the Laurent phenomenon under the cube recurrence, see [8, 11]). Moreover, it turns out that all coefficients in the linear pieces of h are integers between -1 and 2. Also there is an interesting relation between such pieces and special Gelfand-Tsetlin patterns (or semi-standard Young tableaux).

This paper is organized as follows. In Section 2 we prove the simpler part of Theorem A, namely, that the corresponding restriction map res is injective. Also we explain there a rather surprising fact that the TP4-relations follow from the TP3-ones unless $m = m'$. The other part of Theorem A, concerning the surjectivity of res, is more involved. We prove the surjectivity for the Boolean case in Section 3, and for the general case in Section 4, thus completing the proof of Theorem A. Relations between bases, their mutations and rhombic tiling diagrams are discussed in Section 5; here we also consider the problem of extendability of a given subset $X \subset B(a)$ into a normal basis. Sections 6, 7 and 8 are devoted, respectively, to submodular, skew-submodular and discrete concave TP-functions. The concluding Section 9 discusses the tropical Laurentness property for TP-functions.

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2 Injectivity of the restriction map

In this section we prove that the restriction map to the standard basis is injective (which is simpler than the proof of its surjectivity, given throughout Sections 3 and 4) and also demonstrate an additional result.

Consider a truncated box $B = B_m^{m'}(a)$ and let $\mathcal{B} := Sint(a;m) \cup Int(a;m) \cup Int(a;m+1) \cup \ldots \cup I(a;m')$. As in the Introduction, given $x \in B$, we denote the first and last elements in the support $\mathrm{supp}(x)$ of x by $c(x)$ and $d(x)$, respectively. Also we introduce the following numbers:

$$\begin{aligned}
&\alpha(x) \text{ is the maximal } i \in [n] \text{ such that } i < d(x) \text{ and } x_i < a_i; \\
&\beta(x) \text{ is the maximal } i' \in [n] \text{ such that } i' < \alpha(x) \text{ and } x_{i'} > 0; \\
&\gamma(x) \text{ is the maximal } i'' \in [n] \text{ such that } i'' < \beta(x) \text{ and } x_{i''} < a_{i''};
\end{aligned} \tag{5}$$

Observe that $\beta(x)$ does not exist if and only if x is a fint (fuzzy-interval), while $\gamma(S)$ does not exist if and only if x is a fint or a sint (sesquialteral fuzzy-interval).

Proposition 2.1 *The restriction map $res : \mathcal{T}(B) \to \mathbb{R}^{\mathcal{B}}$ is injective, i.e., any TP-function on B is determined by its values within $\mathcal{B}$.*

Proof. Let $f \in \mathcal{T}(B)$ and $x \in B$. When $\beta(x)$ exists, define

$$\eta(x) := |a|(\beta(x) + d(x)) + x_{\beta(x)} + x_{d(x)}, \tag{6}$$

Consider two cases.

Case 1: $|x| = m$. We use induction on η to show that $f(x)$ is determined, via TP4-relations, by the values of f within $Sint(a; m) \cup Int(a; m)$.

If x is a fint or a sint, we are done, so assume this is not the case. Then all numbers $i := \gamma(x)$, $j := \beta(x)$, $k := \alpha(x)$ and $\ell := d(x)$ are well defined, and we have $i < j < k < \ell$. Put $x' := x - 1_j - 1_\ell$ and form five vectors $B := x' + 1_i + 1_k$, $C := x' + 1_i + 1_j$, $D := x' + 1_k + 1_\ell$, $E := x' + 1_i + 1_\ell$ and $F := x' + 1_j + 1_k$. From the definitions in (5) it follows that these vectors belong to B (and have size m). By relation (4) (with x' in place of x), $f(x)$ is computed from the values of f on B, C, D, E, F. Also one can check that each of the latter vectors either is a fint or is a sint or the value of η on it is less than $\eta(x)$.

So we can apply induction on η (the inductive process of computing f on the lowest layer $B_m(a)$ has as a base the family $Sint(a; m) \cup Int(a; m)$).

Case 2: $|x| > m$. We show that $f(x)$ is determined, via TP3-relations, by the values of f within $Sint(a; m) \cup Int(a; m) \cup \ldots \cup Int(a; |x|)$. If x is a fint, we are done, so assume this is not the case. Put $i := \beta(x)$, $j := \alpha(x)$ and $k := d(x)$; then $i < j < k$. Put $x' := x - 1_i - 1_k$. By (3) (with x' in place of x), $f(x)$ is computed via the values of f on the vectors

$$B := x' + 1_j, \quad C := x' + 1_i + 1_j, \quad D := x' + 1_k, \quad E := x' + 1_j + 1_k, \quad F := x' + 1_i$$

(each of which belongs to B, in view of (5) and $|x| > m$). One can check that for each of B, C, D, E, F at least one of the following is true: it is a fint; it belongs to the preceding layer; the value of η on it is less than $\eta(x)$. So we can apply induction on the number of the layer and on η. ■

In the rest of this section we discuss an interrelation between TP3- and TP4-conditions.

Proposition 2.2 *Let f be a function on $B = B_m^{m'}(a)$ and let $m < m'$. Suppose f satisfies all TP3-conditions (3) on B. Then f is a TP-function, i.e. f satisfies the TP4-conditions (4) as well.*

Proof. First we show validity of (4) for a cortege (x, i, j, k, ℓ) with $m < |x| + 2 \leq m'$. We are going to deal with only vectors of the form $x + 1_{i'}$ or $x + 1_{i'} + 1_{j'}$, where

$i', j' \in \{i, j, k, \ell\}$ ($i' \neq j'$). For this reason and to simplify notation, one may assume, w.l.o.g., that $x = \mathbf{0}$ and $(i, j, k, \ell) = (1, 2, 3, 4)$ (in which case we, in fact, deal with the truncated Boolean cube $\{S \subset [4]\colon 1 \leq |S| \leq 2\}$). So we have to prove that

$$f(13) + f(24) = \max\{f(12) + f(34),\ f(14) + f(23)\} \tag{7}$$

(where for brevity qr stands for $1_q + 1_r$).

We use the following three TP3-relations for f:

$$f(24) + f(3) = \max\{f(2) + f(34),\ f(4) + f(23)\}; \tag{8}$$
$$f(13) + f(2) = \max\{f(1) + f(23),\ f(3) + f(12)\}; \tag{9}$$
$$f(14) + f(2) = \max\{f(1) + f(24),\ f(4) + f(12)\}. \tag{10}$$

Adding $f(12)$ to both sides of (8) gives

$$f(24) + f(3) + f(12) = \max\{f(2) + f(34) + f(12),\ f(4) + f(23) + f(12)\}.$$

If in each side of this relation we take the maximum of the expression there and $f(1) + f(23) + f(24)$, then we obtain

$$\begin{aligned} &\max\{f(24) + f(3) + f(12),\ f(1) + f(23) + f(24)\} \\ &= \max\{f(2) + f(34) + f(12),\ f(4) + f(23) + f(12),\ f(1) + f(23) + f(24)\}. \end{aligned}$$

This can be re-written as

$$\begin{aligned} &\max\{f(3) + f(12),\ f(1) + f(23)\} + f(24) \\ &= \max\{f(2) + f(34) + f(12),\ \max\{f(4) + f(12),\ f(1) + f(24)\} + f(23)\}. \end{aligned}$$

The maximum in the left hand side is equal to $f(13)+f(2)$, by (9), and the interior maximum in the right hand side is equal to $f(14) + f(2)$, by (10). Therefore, we have

$$\begin{aligned} f(13) + f(2) + f(24) &= \max\{f(2) + f(34) + f(12),\ f(14) + f(2) + f(23)\} \\ &= \max\{f(34) + f(12),\ f(14) + f(23)\} + f(2). \end{aligned}$$

Canceling out $f(2)$ in the left and right sides, we obtain the required equality (7).

Next, let $|x| + 2 = m$. Take the complementary function $f^*(a - y) := f(y)$, $y \in B_m^{m'}(a)$, and consider the vectors $\overline{x} := x + 1_i + 1_j + 1_k + 1_\ell$ and $\overline{x}^* := a - \overline{x}$. Then f^* is a function on the truncated box $B^* := B_{|a|-m'}^{|a|-m}(a)$ satisfying the TP3-conditions there, and the vector $\overline{x}^*$ is nonnegative and satisfies $\overline{x}^* + 1_i + 1_j + 1_k + 1_\ell \leq a$ and $|a| - m' < |x^*| + 2 = |a| - m$. So we have a situation as in the previous case (w.r.t. B^*) and obtain that relation (4) holds for the cortege $(\overline{x}^*, i, j, k, \ell)$. This implies that (4) holds for f and (x, i, j, k, ℓ). ■

Thus, in the definition of a TP-function, imposing the TP4-relations is important only when we deal with a slice $B_m(a)$.

3 Surjectivity in the Boolean case

In this section we prove the Boolean version of Theorem A.

To distinguish between the Boolean and general cases, we modify notation as follows. Let $0 \le m \le m' \le n$. The parameter n will be fixed throughout and we will usually omit it in notation for basic objects. We denote by $C_m^{m'}$ the truncated Boolean cube $\{S \subseteq [n] \colon m \le |S| \le m'\}$, and by C_p the hyper-simplex consisting of the subsets S of size $|S| = p$ (or the pth layer of $C_m^{m'}$ when $m \le p \le m'$).

Any set $S \subseteq [n]$ is uniquely represented as the union of intervals $I_1 = [c_1..d_1], \dots, I_r = [c_r..d_r]$ such that $c_j > d_{j-1} + 1$ for $j = 2, \dots, r$; such a representation is denoted as

$$S = I_1 \sqcup \dots \sqcup I_r.$$

Recall that a sesquialteral interval, or a *sesqui-interval*, is a set of the form $[1..d_1] \sqcup [c_2..d_2]$ with $d_1 + 1 < c_2$.

We denote the set of p-element intervals in $[n]$ by $\mathcal{I}_p$, and the set of p-element sesqui-intervals by $\mathcal{S}_p$. Then the Boolean version of Theorem A is the following

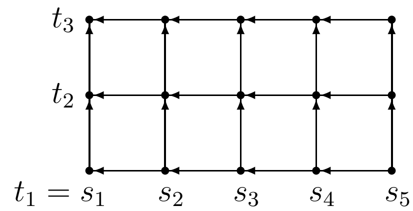
Theorem A$'$ *Let $\mathcal{B} := \mathcal{S}_m \cup \mathcal{I}_m \cup \mathcal{I}_{m+1} \cup \dots \cup \mathcal{I}_{m'}$ and let $\rho : \mathcal{T}(C_m^{m'}) \to \mathbb{R}^{\mathcal{B}}$ be the restriction map. Then ρ is a bijection, i.e., $\mathcal{B}$ forms a TP-basis for the truncated cube $C_m^{m'}$.*

(Note that $\mathcal{S}_m$ vanishes if $m < 2$. Also when $m = 0$, we default assume that $f(\emptyset) = 0$ for any $f \in \mathcal{T}(C_0^{m'})$).

It suffices to prove that ρ in this theorem is surjective, as its injectivity has been shown in the previous section. The proof consists of several steps and is given throughout the subsections below. It is based on a method of generating any TP-function on $C_m^{m'}$ by use of a certain flow model, which we call the *flow-in-matrix method*. This method has as a source a construction of examples of tropical Plücker functions in [1].

3.1 Grids, matrices and flows

By the *grid* of size $n \times m'$ we mean the following directed graph $\Gamma = \Gamma_{n,m'}$. The vertex set $V_{n,m'}$ of Γ consists of elements v_{pq} for $p = 1, \dots, n$ and $q = 1, \dots, m'$. The edge set $E_{n,m'}$ of Γ consists of the pairs $(v_{pq}, v_{p'q'})$ such that either $p' = p - 1$ and $q' = q$, or $p' = p$ and $q' = q + 1$. We visualize the grid by identifying a vertex v_{pq} with the point (p, q) in the plane. Then the vertices $v_{11}, \dots, v_{1,n}$ are located in the bottommost horizontal line of Γ; we call them the *sources* and denote by $s_1, \dots, s_n$, respectively. The vertices $v_{11}, \dots, v_{m',1}$, located in the leftmost vertical line of Γ, are called the *sinks* and denoted by $t_1, \dots, t_{m'}$, respectively. The grid $\Gamma_{5,3}$ is illustrated in the picture.



By a *flow* we mean a collection $\mathcal{F}$ of paths in Γ, each path beginning at a source and ending at a sink. We say that $\mathcal{F}$ is *admissible* if:

(i) the paths in $\mathcal{F}$ are pairwise (vertex) disjoint; and

(ii) the sinks occurring in $\mathcal{F}$ are $t_1, \dots, t_{|\mathcal{F}|}$.

Consider a weighting $w : V_{n,m'} \to \mathbb{R}$ on the vertices. The weight $w(P)$ of a path P is defined to be the sum of weights $w(v)$ of the vertices v of P, and the weight $w(\mathcal{F})$ of a flow $\mathcal{F}$ is $\sum(w(P) \colon P \in \mathcal{F})$. For a set $S \subseteq [n]$ with $|S| \le m'$, define

$$f_w(S) := \max\{w(\mathcal{F})\}, \tag{11}$$

where the maximum is taken over all admissible flows $\mathcal{F}$ in $\Gamma_{n,m'}$ beginning at the set $\{s_p \colon p \in S\}$.

In what follows we will identify the weighting w with the $n \times m'$ matrix $W = (w_{pq})$, where $w_{pq} = w(v_{pq})$. To be consistent with the visualization of the grid, we should think of n as the number of columns of W, use the first index just for the columns, and assume that w_{11} is the leftmost and bottommost element of the matrix.

The following assertion plays the key role in our proof.

Theorem 3.1 *Let $W = (w_{pq})$ be a real $n \times m'$ matrix. Then the function f_w defined by (11) on the sets $S \in C_m^{m'}$ is a TP-function.*

We denote the map $W \mapsto f_w$ in this proposition by ϕ. It should be noted that ϕ is not injective in general, that is, one and the same function f may be derived from several matrices. In order to get a one-to-one correspondence, we will restrict the set of matrices by taking as W a matrix that is the sum of two $n \times m'$ matrices W', W'', where

(12) W' is such that $w'_{pq} = 0$ for all p, q with $p < \max\{q, m+1\}$;

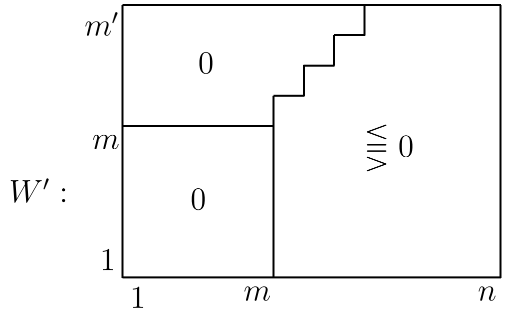
(13) W'' is such that:

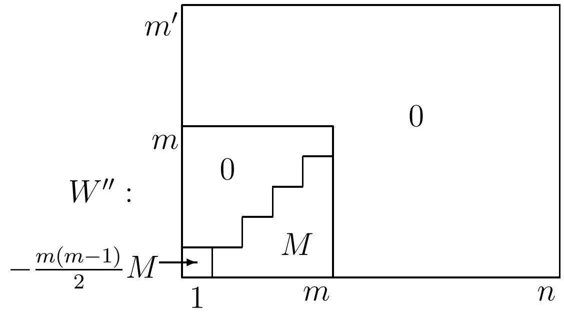
(i) $w''_{pq} = M$ for $p = 2, \dots, m$ and $q < p$;

(ii) $w''_{11} = -\frac{m(m-1)}{2} M$;

(iii) $w''_{pq} = 0$ otherwise.

Here M is a sufficiently large positive number w.r.t. the entries of W' (one can take $M := nm \max\{|w'_{pq}|\}$). The purpose of adding the matrix W'' to W' will be clear later. The behavior of W', W'' is illustrated in the picture.





(Note that when $m < 2$, W'' becomes the zero matrix and $W = W'$. When $m = 0$ and $m' = n$, the essential part of W' is a triangular matrix. When $m = m'$, the essential part of W' is an $(n-m) \times n$ matrix.) We denote the sets of matrices W and W' as above by $\mathcal{W} = \mathcal{W}(m, m')$ and $W' = \mathcal{W}'(m, m')$, respectively (W is considered up to the choice of M).

We say that a function f on $C_m^{m'}$ or on $\mathcal{B}$ is *normalized* if $f([1..m]) = 0$. The set of normalized functions on $C_m^{m'}$ is denoted by $\mathcal{T}^0(C_m^{m'})$, and we denote by $\mathcal{B}^0$ the set $\mathcal{B}$ from which the interval $\{[1..m]\}$ is removed. (Any TP-function can be considered up to adding a constant; so in Theorem A$'$ one can consider only normalized TP-functions and their restrictions to $\mathcal{B}$.) The importance of (12),(13) is emphasized by the following

Proposition 3.2 *For each normalized function $g : \mathcal{B} \to \mathbb{R}$, there exists a matrix $W' \in \mathcal{W}'(m, m')$ such that $g(S) = f_w(S)$ holds for all $S \in \mathcal{B}$, where the weighting w corresponds to $W = W' + W''$. Moreover, W' is unique and the correspondence of g and W' gives a bijection between $\mathbb{R}^{\mathcal{B}^0}$ and $\mathcal{W}'(m, m')$, or between $\mathbb{R}^{\mathcal{B}^0}$ and $\mathcal{W}(m, m')$ (considering the matrices in $\mathcal{W}(m, m')$ up to the choice of M)).*

We denote the map $g \mapsto W$ in this proposition by μ.

Summing up Theorem 3.1 and Proposition 3.2, we can conclude that the restriction map ρ is surjective. Indeed, for each normalized function g on $\mathcal{B}$, take the matrix $W = \mu(g)$ and form the function $f = f_w$ $(= \phi(W))$ on $C_m^{m'}$. Then f is a TP-function whose restriction to $\mathcal{B}$ is just g. In fact, we have three bijections.

Corollary 3.3 *The maps ρ, μ, ϕ determine bijections between $\mathcal{T}^0(C_m^{m'})$ and $\mathbb{R}^{\mathcal{B}^0}$, between $\mathbb{R}^{\mathcal{B}^0}$ and $\mathcal{W}(m, m')$, and between $\mathcal{W}(m, m')$ and $\mathcal{T}^0(C_m^{m'})$, respectively. Their composition $\phi \circ \mu \circ \rho$ is identical on $\mathcal{T}^0(C_m^{m'})$. (See the picture.)*

$$\begin{tikzcd} \mathcal{T}^0 \arrow[r, "\mathrm{id}"] \arrow[d, "\rho"'] & \mathcal{T}^0 \\ \mathbb{R}^{\mathcal{B}^0} \arrow[r, "\mu"'] & \mathcal{W} \arrow[u, "\phi"'] \end{tikzcd}$$

Thus, it remains to prove Proposition 3.2 and Theorem 3.1.

3.2 From functions on $\mathcal{B}$ to matrices

In this subsection we prove Proposition 3.2.

For an $n \times m'$ matrix $\widetilde{W}$ and subsets $I \subseteq [n]$ and $J \subseteq [m']$, let $\widetilde{w}(I \times J)$ denote the weight $\sum(\widetilde{w}_{pq} \colon p \in I,\ q \in J)$ of the $I \times J$ submatrix of $\widetilde{W}$.

Let g be a normalized function on $\mathcal{B}$. The desired matrix W' for g is assigned so as to satisfy the following conditions:

$$g([c..d]) = w'([d] \times [d-c+1]) \quad \text{for each interval } [c..d] \in \mathcal{B}; \tag{14}$$

$$g(S) = w'([d_2] \times [d_2 - c_2 + 1]) \quad \text{for each sint } S = [1..d_1] \sqcup [c_2..d_2] \in \mathcal{B}. \tag{15}$$

Subject to (12), these conditions determine W' uniquely. To see this, let $\Pi := \{(p,q) \colon p = m+1, \dots, n,\ q = 1, \dots, \min\{p, m'\}\}$ (the set of essential index pairs in (12)). There is a natural bijection $\pi : \Pi \to \mathcal{B}^0$, namely:

(16) for $(p,q) \in \Pi$,

(i) $\pi(p,q)$ is the interval $[p-q+1..p]$ (of size q) if $q \ge m$;

(ii) $\pi(p,q)$ is the sint $[1..m-q] \sqcup [p-q+1..p]$ (of size m) if $q < m$.

Now using (14)–(15), one can compute the weights w'_{pq} for all $(p,q) \in \Pi$, step-by-step by increasing p,q; they are expressed as

$$\begin{array}{rcll}
w'_{m+1,1} & = & g(\pi(m+1,1)); & \\
w'_{p,1} & = & g(\pi(p,1)) - g(\pi(p-1,1)) & \text{if } p > m+1; \\
w'_{pq} & = & g(\pi(p,q)) - g(\pi(p,q-1)) & \text{if } p = \max\{q, m+1\} \text{ and } q > 1; \\
w'_{pq} & = & \multicolumn{2}{l}{g(\pi(p,q)) + g(\pi(p-1,q-1)) - g(\pi(p-1,q)) - g(\pi(p,q-1))} \\
 & & \qquad\qquad \text{otherwise.} &
\end{array} \tag{17}$$

(This will also be used in Section 9.) Thus, (14)–(15) (or (17)) gives a bijection between $\mathcal{W}'(m,m')$ and $\mathbb{R}^{\mathcal{B}^0}$.

We assert that the matrix $W = W' + W''$ is as required in Proposition 3.2 for the given g, i.e., $g(S) = f_w(S)$ holds for all intervals and sesqui-intervals $S \in \mathcal{B}$.

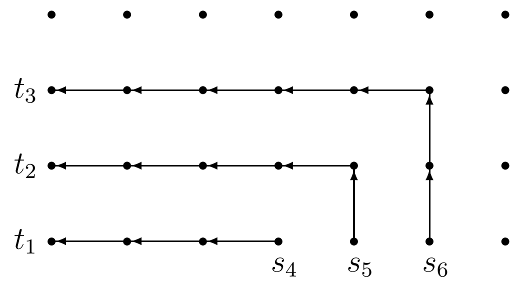
To show this, first of all observe that the matrix W'' in (13) satisfies

$$w''([p] \times [q]) = 0 \qquad \text{if } p,q \ge m. \tag{18}$$

Consider an interval $I = [c..d] \in \mathcal{B}$. In the grid $\Gamma = \Gamma_{n,m'}$ there is a unique admissible flow $\mathcal{F}$ having the source set $\{s_p \colon p \in [c..d]\}$. This flow consists of the paths $P_1, \dots, P_{d-c+1}$, where each P_i begins at the source $s_{\bar{i}}$ for $\bar{i} := c+i-1$, ends at the sink t_i, and is of the form

$$P_i = (s_{\bar{i}} = v_{\bar{i},1}, v_{\bar{i},2}, \dots, v_{\bar{i},i}, v_{\bar{i}-1,i}, \dots, v_{1,i} = t_i).$$

(Hereinafter we use notation for a path without indicating its edges.) The picture below illustrates $\mathcal{F}$ in the case $c = 4$ and $d = 6$.



So the function f generated by W via the flow model satisfies

$$f(I) = w([d] \times [d-c+1]) = w'([d] \times [d-c+1]) = g(I)$$

(in view of (18) and $|I| \geq m$).

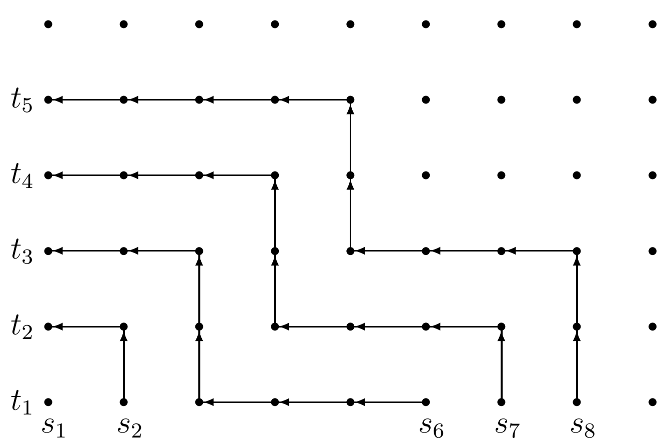
Next consider a sesqui-interval $S = [1..d_1] \sqcup [c_2..d_2]$ in $\mathcal{B}$. We associate to a path P from a source s_i to a sink t_j in Γ the closed region of the plane bounded by P, by the horizontal path from s_i to v_{11} and by the vertical path from v_{11} to t_j; we call it the *lower region* of P and denote by $\mathcal{R}(P)$. (Regions of this sort will be used in the next subsection as well.)

In contrast to intervals, the sesqui-interval S admits several admissible flows with the source set $\{s_p \colon p \in S\}$. We distinguish one flow $\mathcal{F}$ among these; it is called the *lowest* flow for S and consists of the paths $P_1, \ldots, P_m$ whose lower regions are as small as possible.

These paths are constructed as follows. For $i = 1, \ldots, d_1$, the path P_i (going from s_i to t_i) can be chosen uniquely, namely, P_i is $(v_{i,1}, \ldots, v_{i,i}, \ldots, v_{1,i})$. Now let $d_1 < i \leq m$. Put $i' := i - d_1$ and $\overline{i} := c_2 + i' - 1$; then $\overline{i} \in [c_2..d_2]$. One can see that the path P_i, that goes from $s_{\overline{i}}$ to t_i and has the minimal lower region (provided that $P_1, \ldots, P_{i-1}$ are already constructed), is of the form

$$P_i = (v_{\overline{i},1}, \ldots, v_{\overline{i},i'}, \ldots, v_{d_1+i',i'}, \ldots, v_{d_1+i',i}, \ldots, v_{1,i})$$

(as a rule, P_i makes three turns). An instance of a lowest flow is illustrated in the picture; here $m = 5$, $d_1 = 2$, $c_2 = 6$ and $d_2 = 8$.



Observe that the vertex set of the lowest flow $\mathcal{F}$ for the given S spans the region being the union of the square $[1..m]\times[1..m]$ and the rectangle $[m+1..d_2]\times[1..d_2-c_2+1]$. So, in view of (18), the weight $w(\mathcal{F})$ of $\mathcal{F}$ amounts to $w'([m+1..d_2]\times[1..d_2-c_2+1])$, which is equal to $w'([1..d_2]\times[1..d_2-c_2+1])=g(S)$ (cf. (15)). We assert that $\mathcal{F}$ is the maximum-weight flow for w and S.

Indeed, it is not difficult to see that any other admissible flow $\mathcal{F}'$ for S does not cover at least one vertex v_{pq} with $q<p\le m$. This means that the total contribution to $w(\mathcal{F}')$ from the vertices v_{pq} such that $1\le p,q\le m$ is at most $-M$, whereas a similar contribution for $\mathcal{F}$ is zero. Since M is large, we obtain $w(\mathcal{F}')<w(\mathcal{F})$ (this is just where the matrix W'' is important). So $f_w(S)=w(\mathcal{F})=g(S)$, as required.

This completes the proof of Proposition 3.2.

3.3 Rearranging flows in the grid

Now we start proving Theorem 3.1 claiming that the function f on $C_m^{m'}$ generated by use of the flow model from any real $n\times m'$ matrix W is indeed a TP-function. In this subsection we prove a weakened version of this theorem. It is stated in two lemmas (cf. equalities (1) and (2)).

Lemma 3.4 *For an $n\times m'$ matrix W and the function $f=f_w$ on $C_0^{m'}$,*

$$f(Xik)+f(Xj)\ge\max\{f(Xij)+f(Xk),\ f(Xi)+f(Xjk)\} \qquad (19)$$

holds for all $i<j<k$ and $X\subset[n]-\{i,j,k\}$ with $|X|\le m'-2$.

Proof. We essentially use the fact that the grid $\Gamma=\Gamma_{n,m'}$ is a planar graph. Let $r:=|X|+2$. W.l.o.g., we may assume that $f(Xij)+f(Xk)\ge f(Xi)+f(Xjk)$. Let $\mathcal{F}=\{P_1,\ldots,P_r\}$ be an admissible flow in Γ for Xij (i.e., going from the sources $\{s_p\colon p\in X\}\cup\{s_i,s_j\}$ to the sinks $\{t_1,\ldots,t_r\}$) such that $f(Xij)=w(\mathcal{F})$ (cf. (11)), and $\mathcal{F}'=\{P'_1,\ldots,P'_{r-1}\}$ an admissible flow for Xk such that $f(Xk)=w(\mathcal{F}')$.

We combine these flows into one family $\mathcal{P}=\{P_1,\ldots,P_r,P'_1,\ldots,P'_{r-1}\}$ (possibly containing repeated paths). Observe that

(i) each vertex of Γ belongs to at most two paths in $\mathcal{P}$;

(ii) for $p\in[n]$, the source s_p is the beginning of exactly one path in $\mathcal{P}$ if $p\in\{i,j,k\}$, and the beginning of exactly two paths if $p\in X$;

(iii) each of the sinks $t_1,\ldots,t_{r-1}$ is the end of exactly two paths in $\mathcal{P}$, and t_t is the end of exactly one path.

Using a standard planar flow decomposition technique, one can rearrange paths in $\mathcal{P}$ so as to obtain a family $\mathcal{Q}=\{Q_1,\ldots,Q_{2r-1}\}$ of paths from sources to sinks having properties (ii),(iii) as above (with $\mathcal{Q}$ in place of $\mathcal{P}$), and in addition:

(iv) for each vertex v of Γ, the numbers of occurrences of v in paths of $\mathcal{Q}$ and in paths of $\mathcal{P}$ are equal;

(v) $\mathcal{R}(Q_1)\subseteq\mathcal{R}(Q_2)\subseteq\ldots\subseteq\mathcal{R}(Q_{2r-1})$,

where $\mathcal{R}(P)$ is the lower region of a path P, defined in Subsection 3.2. (Such a $\mathcal{Q}$ is constructed uniquely.) Partition $\mathcal{Q}$ into two subfamilies:

$$\mathcal{F}_1 := \{Q_p \colon p \text{ is odd}\} \quad \text{and} \quad \mathcal{F}_2 := \{Q_p \colon p \text{ is even}\} \tag{20}$$

We assert that each of these subfamilies consists of pairwise disjoint paths. Indeed, suppose this is not so. Then, in view of (v), some subfamily contains “consecutive” paths Q_p, Q_{p+2} that share a common vertex v. But now the inclusions $\mathcal{R}(Q_p) \subseteq \mathcal{R}(Q_{p+1}) \subseteq \mathcal{R}(Q_{p+2})$ imply that v must belong to the third path Q_{p+1} as well, which is impossible by (i) and (iv).

This assertion together with (ii),(iii),(iv) easily implies that both $\mathcal{F}_1, \mathcal{F}_2$ are admissible flows, that the set of the beginning vertices of paths in $\mathcal{F}_1$ consists of the sources s_i,s_k and s_p for all $p \in X$, and that the set of the beginning vertices of paths in $\mathcal{F}_2$ consists of the sources s_j and s_p for all $p \in X$. Here we use the fact that, due to $i < j < k$, the paths in $\mathcal{Q}$ beginning at s_i, s_j, s_k have odd, even and odd indices, respectively.

Thus, $\mathcal{F}_1$ and $\mathcal{F}_2$ are admissible flows for Xik and Xj, respectively. Also (iv) implies $w(\mathcal{F}_1) + w(\mathcal{F}_2) = w(\mathcal{F}) + w(\mathcal{F}')$. By the definition of f, we have $f(Xik) \geq w(\mathcal{F}_1)$ and $f(Xj) \geq w(\mathcal{F}_2)$, and (19) follows. ■

Lemma 3.5 *For an $n \times m'$ matrix W and the function $f = f_w$ on $C_0^{m'}$,*

$$f(Xik) + f(Xj\ell) \geq \max\{f(Xij) + f(Xk\ell),\ f(Xi\ell) + f(Xjk)\} \tag{21}$$

holds for all $i < j < k < \ell$ and $X \subset [n] - \{i, j, k, \ell\}$ with $|X| \leq m' - 2$.

Proof. It is similar to the above proof. Assume $f(Xij) + f(Xk\ell) \geq f(Xi\ell) + f(Xjk)$ (when the reverse inequality holds, the method is similar). Take in $\Gamma_{n,m'}$ an admissible flow $\mathcal{F}$ for Xij such that $f(Xij) = w(\mathcal{F})$, and an admissible flow $\mathcal{F}'$ for $Xk\ell$ such that $f(Xk\ell) = w(\mathcal{F}')$. Combine these into one family $\mathcal{P}$, and rearrange paths in $\mathcal{P}$ so as to obtain a (unique) family $\mathcal{Q} = \{Q_1, \ldots, Q_{2m}\}$ of paths from sources to sinks satisfying (iv),(v) as in the above proof (with $2m$ in place of $2r-1$). Partition $\mathcal{F}$ into subfamilies $\mathcal{F}_1, \mathcal{F}_2$ as in (20). Then $i < j < k < \ell$ implies that $\mathcal{F}_1$ is an admissible flow for Xik, and $\mathcal{F}_2$ is an admissible flow for $Xj\ell$, whence (21) follows. ■

3.4 Getting equalities (1)–(2)

It remains to show that the inequalities reverse to (19),(21) are valid as well, i.e.,

$$f(Xik) + f(Xj) \leq \max\{f(Xij) + f(Xk),\ f(Xi) + f(Xjk)\} \tag{22}$$

and

$$f(Xik) + f(Xj\ell) \leq \max\{f(Xij) + f(Xk\ell),\ f(Xi\ell) + f(Xjk)\}, \tag{23}$$

hold for a function f derived from an $n \times m'$ matrix W and for corresponding X, i, j, k, ℓ. In fact, these relations can be obtained from a result in [15, p. 60]

(whereas (19),(21) cannot). To make our description self-contained, we give direct proofs of (22) and (23) here, by arguing in a similar spirit as in [15].

To prove (22), we take in $\Gamma = \Gamma_{n,m'}$ an admissible flow $\mathcal{F}$ for Xik with $w(\mathcal{F}) = f(Xik)$, and an admissible flow $\mathcal{F}'$ for Xj with $w(\mathcal{F}') = f(Xj)$. Regarding $\mathcal{F}$ as a graph, we modify it as follows. Each vertex v of $\mathcal{F}$ is replaced by edge $e_v = (v', v'')$; each original edge (u, v) of $\mathcal{F}$ is transformed into edge (u'', v'). The resulting graph, consisting of pairwise disjoint paths as before, is denoted by $\gamma(\mathcal{F})$. The graph $\mathcal{F}'$ is modified into $\gamma(\mathcal{F}')$ in a similar way. Corresponding edges of $\gamma(\mathcal{F})$ and $\gamma(\mathcal{F}')$ are identified.

Next we construct an auxiliary graph H by the following rule:

(a) if e is an edge in $\gamma(\mathcal{F})$ but not in $\gamma(\mathcal{F}')$, then e is included in H;

(b) if $e = (u, v)$ is an edge in $\gamma(\mathcal{F}')$ but not in $\gamma(\mathcal{F})$, then the edge (v, u) *reverse* to e is included in H.

(Common edges of $\gamma(\mathcal{F}), \gamma(\mathcal{F}')$ are not included in H.) One can see that H has the following properties: each vertex has at most one incoming edge and at most one outgoing edge; the vertices having one outgoing edge and no incoming edge are exactly s'_i, s'_k; the vertices having one incoming edge and no outgoing edge are exactly s'_j, t''_r, where $r = |X| + 2$. This implies that H is represented as the disjoint union of cycles, isolated vertices and two paths P, Q, where either P is a path from s'_i to s'_j and Q is a path from s'_k to t''_r (*Case 1*), or P is a path from s'_k to s'_j and Q is a path from s'_i to t''_r (*Case 2*).

We use the path P to rearrange the graphs $\gamma(\mathcal{F})$ and $\gamma(\mathcal{F}')$ as follows: for each edge $e = (u, v)$ of P,

(c) if e is in $\gamma(\mathcal{F})$, then we delete e from $\gamma(\mathcal{F})$ and add to $\gamma(\mathcal{F}')$;

(d) if e is not in $\gamma(\mathcal{F})$, and therefore, the edge $\overline{e} = (v, u)$ reverse to e is in $\gamma(\mathcal{F}')$, then we delete $\overline{e}$ from $\gamma(\mathcal{F}')$ and add to $\gamma(\mathcal{F})$.

Let $\mathcal{G}$ and $\mathcal{G}'$ be the graphs obtained in this way from $\gamma(\mathcal{F})$ and $\gamma(\mathcal{F}')$, respectively (if there appear isolated vertices, we ignore them). In these graphs we shrink each edge of the form $e_v = (v', v'')$ into one vertex v. This produces subgraphs $\mathcal{F}_1$ and $\mathcal{F}_2$ of Γ, where the former corresponds to $\mathcal{G}$, and the latter to $\mathcal{G}'$.

It is not difficult to deduce from (a)–(d) that each of $\mathcal{F}_1, \mathcal{F}_2$ consists of pairwise disjoint paths, and moreover: in Case 1, $\mathcal{F}_1$ is an admissible flow for Xjk and $\mathcal{F}_2$ is an admissible flow for Xi, while in Case 2, $\mathcal{F}_1$ is an admissible flow for Xij and $\mathcal{F}_2$ is an admissible flow for Xk. Also one can see that for each vertex v of Γ, the numbers of occurrences of v in paths of $\{\mathcal{F}_1, \mathcal{F}_2\}$ and in paths of $\{\mathcal{F}, \mathcal{F}'\}$ are the same. Therefore, $w(\mathcal{F}) + w(\mathcal{F}') = w(\mathcal{F}_1) + w(\mathcal{F}_2) \le f(Xij) + f(Xk)$, yielding (22).

Finally, (23) is shown by a similar method, and we give here only a short outline, leaving details to the reader. Take in Γ an admissible flow $\mathcal{F}$ for Xik with $w(\mathcal{F}) = f(Xik)$, and an admissible flow $\mathcal{F}'$ for $Xj\ell$ with $w(\mathcal{F}') = f(Xj\ell)$. Construct the corresponding graphs $\gamma(\mathcal{F})$, $\gamma(\mathcal{F}')$ and H. Then H is represented as the disjoint union of cycles, isolated vertices and two paths P, Q, where either P is a path from s'_i to s'_j and Q is a path from s'_k to s'_ℓ (*Case 1′*), or P is a path from s'_k to s'_j and Q is a path from s'_i to s'_ℓ (*Case 2′*). Using the path P, we rearrange $\gamma(\mathcal{F}), \gamma(\mathcal{F}')$

according to (c),(d), eventually obtaining subgraphs $\mathcal{F}_1, \mathcal{F}_2$ of Γ. Then both $\mathcal{F}_1, \mathcal{F}_2$ are admissible flows. Furthermore, in Case $1'$, $\mathcal{F}_1$ is a flow for Xjk and $\mathcal{F}_2$ is a flow for $Xi\ell$, while in Case $2'$, $\mathcal{F}_1$ is a flow for Xij and $\mathcal{F}_2$ is a flow for $Xk\ell$. Also $w(\mathcal{F}_1) + w(\mathcal{F}_2) = w(\mathcal{F}) + w(\mathcal{F}')$, and (23) follows.

This gives Theorem 3.1 and completes the proof of Theorem A$'$. ■■

Remark 1. As is said in the Introduction, Theorem A implies that any TP-function on a truncated box can be extended to a TP-function on the entire box. In the Boolean case, this property can also be immediately seen from the flow-in-matrix method. Indeed, given a TP-function f on the truncated cube $C_m^{m'}$, take an $n \times m'$ matrix W determining f and extend W arbitrarily into an $n \times n$ matrix W'. Then W' generates the desired TP-extension of f to the Boolean cube $2^{[n]}$.

4 Surjectivity in the general case

In this section we complete the proof of Theorem A in the general case, by showing that the corresponding restriction map *res* is surjective. We will use a reduction to the Boolean case and Theorem A$'$.

4.1 Surjectivity in the case of an entire box.

We start with considering an arbitrary truncated box $B_m^{m'}(a)$. For $i = 1, \ldots, n$, denote $a_1 + \ldots + a_i$ by $\overline{a}_i$, and let $N := \overline{a}_n = |a|$. The ordered set $[N]$ is naturally partitioned into intervals (*blocks*) $L_1, \ldots, L_n$, where $L_i := \overline{a}_{i-1} + [a_i] = [\overline{a}_{i-1} + 1..\overline{a}_i]$ (letting $\overline{a}_0 := 0$). We associate to a vector $x \in B_m^{m'}(a)$ the subset $[x]$ of $[N]$ such that

(24) for $i = 1, \ldots, n$, the set $[x] \cap L_i$ consists of x_i *beginning* elements of L_i (i.e., is of the form $\overline{a}_{i-1} + [x_i]$).

This gives the map $[\,] : B_m^{m'}(a) \to C_m^{m'}(N)$, where $C_m^{m'}(N)$ is the truncated cube formed by the sets $S \subseteq [N]$ with $m \le |S| \le m'$. Conversely,

(25) for $S \subseteq [N]$, define $\#(S)$ to be the vector $x \in B(a)$ such that $x_i = |S \cap L_i|$, $i = 1, \ldots, n$.

This gives the map $\# : C_m^{m'}(N) \to B_m^{m'}(a)$. Observe that $\#([x]) = x$. The map $[\,]$ induces the corresponding map $[\,]^*$ of the functions g on $C_m^{m'}(N)$ to functions f on $B_m^{m'}(a)$: the value of f on $x \in B_m^{m'}(a)$ is equal to $g([x])$. We observe that

(26) $[\,]^*(g)$ brings a TP-function g on $C_m^{m'}(N)$ to a TP-function f on $B_m^{m'}(a)$.

Indeed, one can see that for the six vectors x' occurring in relation (3), the corresponding sets $[x']$ are just those that occur in (1). So validity of (1) for the latter implies validity of (3) for the former. For (4) and (2), the argument is similar.

In light of (26), the surjectivity of *res* would follow from the assertion:

(27) for any function $f_0 : \mathcal{B} \to \mathbb{R}$, there is a TP-function g on $C_m^{m'}(N)$ such that $g([x]) = f_0(x)$ holds for all $x \in \mathcal{B}$.

(Indeed, the image by $[\,]^*$ of g is a TP-function f on $B_m^{m'}(a)$ coinciding with f_0 within $\mathcal{B}$, i.e., having the required property $res(f) = f_0$.)

In the rest of this subsection we prove (27) in the case when the box is truncated only from above, i.e., when $m = 0$ or, equivalently, $m = 1$ (this is technically simpler because in this case $\mathcal{B}$ does not contain sints). Using this as a base, we will prove, in the next subsection, the surjectivity of res for any m by applying induction on m.

Proposition 4.1 *(27) is valid for $\mathcal{B} := Int(a;1) \cup \ldots \cup I(a;m')$ and $C_0^{m'}(N)$.*

Proof. By Theorem A$'$, it is sufficient to assign, in a due way, values of g on the set of intervals I of $C_0^{m'}(N)$. If $[\#(I)] = I$, i.e., if I is the image by $[\,]$ of a fuzzy-interval, the task is trivial: we put $g(I) = f_0(\#(I))$. But when $[\#(I)] \neq I$, our method of assigning $g(I)$ via $f_0(\#(I))$ becomes more involved. Moreover, to prove the correctness of our assignment, we will be forced to express g explicitly for a larger family of sets.

To explain the idea, consider a fint x and let $[c..d]$ be its support. If $c = d$ or if $x_c = a_c$, then $[x]$ is an interval in $[N]$. In a general case we partition $[x]$ into two subsets: the *head* $H(x) := [x] \cap L_c$ and the *tail* $T(x) := [x] \cap [\overline{a}_c + 1..N]$. Clearly both the head and the tail are intervals (unless $T(x) = \emptyset$).

Besides $[x]$, we produce from x additional sets $Q_1, \ldots, Q_q$ in $[N]$, where $q = a_c - x_c$ and for $p = 1, \ldots, q$, Q_p is obtained from $[x]$ by shifting its head $H(x)$ by p positions to the right. Formally: $Q_p := (p + H(x)) \cup T(x)$. The last set Q_q (whose head is pressed to the end of the block L_c) is already a genuine interval. We call each of $Q_0 := [x], Q_1, \ldots, Q_q$ a *quasi-interval*, and associate to each $Q = Q_p$ three numbers: $h(Q) := x_c$ (the size of the head), $s(Q) := p$ (the *shift* of the head), and $\epsilon(Q) := s(Q)h(Q)$ (the *excess* of Q). Notice that the excess of $[x]$ is zero, whereas the excess of the interval Q_q is maximal among the quasi-intervals created from x.

Let $\mathcal{Q}$ be the family of all quasi-intervals in $C_0^{m'}(N)$ (in particular, $\mathcal{Q}$ contains all intervals and the images of all fints). This set is just where we indicate g explicitly, as follows. Choose a large positive number M (w.r.t. f_0). Define

$$g(Q) := f_0(\#(Q)) + M\epsilon(Q) \qquad \text{for each } Q \in \mathcal{Q}. \tag{28}$$

Then g satisfies the equalities in (27). It remains to prove the following

Claim. *Let g be the function on $\mathcal{Q}$ defined by (28). Then g is extendable to a TP-function on $C_0^{m'}(N)$.*

Proof of the Claim. Let g' be the TP-function on $C_0^{m'}(N)$ coinciding with g on the set of intervals (the standard basis there). We show that g and g' are equal on all quasi-intervals as well.

Consider a quasi-interval Q which is not an interval, and let $i = i(Q)$ and $k = k(Q)$ be the first and last elements of Q, respectively. We apply induction on

$r(Q) := k(Q) - i(Q)$. Take the element $j \in [N]$ next to the end of the head of Q. Then $j \notin Q$ and $i < j < k$. Form the sets $X := Q - \{i, k\}$ and

$$B := Xj, \quad C := Xij, \quad D := Xk, \quad E := Xjk, \quad F := Xi.$$

These five sets Q' are quasi-intervals (maybe even intervals) with $r(Q') < r(Q)$. So, by induction, $g'(Q') = g(Q')$ holds for these Q'. Let h, s, ϵ stand for $h(Q), s(Q), \epsilon(Q)$, respectively. A direct verification shows that

$$\begin{aligned}
&\epsilon(Q) = sh;\\
&\epsilon(B) = (s+1)h \quad \text{(the head moves to the right);}\\
&\epsilon(C) = s(h+1) \quad \text{(the head increases at the end);}\\
&\epsilon(D) = (s+1)(h-1) \quad \text{(the head decreases at the beginning);}\\
&\epsilon(E) = (s+1)h \quad \text{(the head moves to the right);}\\
&\epsilon(F) = sh \quad \text{(the head is stable).}
\end{aligned}$$

(Note that when $h = 1$, we have $h(D) \le a_{i+1}$ and $s(D) = 0$, so the expression for $\epsilon(D)$ gives the correct value 0.) It follows that

$$\epsilon(Q) + \epsilon(B) = \epsilon(E) + \epsilon(F) > \epsilon(C) + \epsilon(D). \tag{29}$$

Since M is large, we obtain from (28) and (29) that $g(E)+g(F) > g(C)+g(D)$. Therefore,

$$g'(Q) + g(B) = g(E) + g(F), \tag{30}$$

taking into account that g' is a TP-function and that, by induction, g' and g are equal on B, C, D, E, F. Next, observe that $\#(Q) = \#(E)$ and $\#(B) = \#(F)$. Therefore, $f_0(\#(Q)) + f_0(\#(B)) = f_0(\#(E)) + f_0(\#(F))$. This together with the equality in (29) gives

$$g(Q) + g(B) = g(E) + g(F).$$

Now comparing this and (30), we conclude that $g'(Q) = g(Q)$. This yields the Claim and completes the proof of the proposition. ■ ■

4.2 Reduction to the entire box.

To complete the proof of Theorem A, it remains to show the surjectivity of the restriction map *res* for an arbitrary m. We apply induction on m, relying on Proposition 4.1 which gives a base for the induction.

Proposition 4.2 *Let $0 < m \le m'$ and let the restriction map $res' : \mathcal{T}(B^{m'}_{m-1}(a)) \to \mathbb{R}^{\mathcal{B}'}$ be surjective, where $\mathcal{B}' = Sint(a; m-1) \cup Int(a; m-1) \cup Int(a; m) \cup \ldots \cup I(a; m')$. Then $res : \mathcal{T}(B^{m'}_{m}(a)) \to \mathbb{R}^{\mathcal{B}}$ is surjective as well.*

Proof. Let f_0 be a function on $\mathcal{B}$. Our aim is to construct a function g_0 on $\mathcal{B}'$ satisfying the following conditions:

(a) g_0 and f_0 are equal on the set $\mathcal{D} := Int(a; m) \cup \ldots \cup Int(a; m')$; and

(b) the TP-function g on $B^{m'}_{m-1}(a)$ with $res'(g) = g_0$ satisfies

$$g(x) = f_0(x) \qquad \text{for each } x \in Sint(a; m). \tag{31}$$

Then (a),(b) imply that the restriction f of g to $B^{m'}_m(a)$ is a TP-function possessing the desired property $res(f) = f_0$.

The desired function g_0 is defined on the vectors in $\mathcal{B}' - \mathcal{D} = Sint(a; m-1) \cup Int(a : m-1)$ as follows.

For $y \in \mathcal{B}' - \mathcal{D}$, let $p = p(y)$ denote the minimal number such that $y_p < a_p$. We refer to $p(y)$ as the *insertion point* for y and denote the vector $y + 1_p$ by $y^\uparrow$. This $y^\uparrow$ has size m and lies in $B^{m'}_m(a)$. Moreover, $y^\uparrow$ is either a fint or a sint. Define

$$g_0(y) := f_0(y^\uparrow) + Mt(y),$$

where M is a large positive number (w.r.t. f_0) and $t(y) := y_{p+1} + \ldots + y_n$.

We assert that g_0 defined this way satisfies (31). To show this, consider $x \in Sint(a; m)$. Let $\alpha(x), \beta(x)$ be defined as in (5) (they exist, otherwise x would be a fint), and assign the parameter $\eta(x)$ as in (6). Put $i := \beta(x)$, $j := \alpha(x)$ and $k = d(x)$. By the TP3-relation for the function g and the cortege $(x - 1_i - 1_k, i, j, k)$, we have

$$g(x) = \max\{g(C) + g(D), g(E) + g(F)\} - g(B), \tag{32}$$

where $B := x - 1_i + 1_j - 1_k$, $C := x + 1_j - 1_k$, $D := x - 1_i$, $E := x - 1_i + 1_j$, $F := x - 1_k$. We observe the following, letting $\Sigma := x_{i+1} + \ldots + x_n$.

(i) The vectors C and E have size m, C is either a fint or a sint with $\eta(C) < \eta(x)$, and similarly for E. So, applying induction on η, we have $g(C) = f_0(C)$ and $g(E) = f_0(E)$.

(ii) The vector B has size $m-1$ and its insertion point is i. Then $B^\uparrow = B + 1_i = C$. Also $t(B) = \Sigma + 1 - 1 = \Sigma$, whence $g(B) = f_0(C) + M\Sigma$.

(iii) The vector D has size $m-1$ and its insertion point is i. Then $D^\uparrow = D + 1_i = x$. Also $t(D) = \Sigma$, whence $g(D) = f_0(x) + M\Sigma$.

(iv) The vector F has size $m-1$ and its insertion point is at least i. This and $F_k = x_k - 1$ imply $t(F) \le \Sigma - 1$, whence $g(F) \le f_0(F^\uparrow) + M\Sigma - M$.

Since M is large and $t(D) \ge t(F) + M$ (by (iii),(iv)), the maximum in (32) is attained by the first sum occurring there. Therefore, in view of (i)–(iii),

$$g(x) = g(C) + g(D) - g(B) = f_0(C) + (f_0(x) + M\Sigma) - (f_0(C) + M\Sigma) = f_0(x),$$

as required, yielding the proposition. ■

This completes the proof of Theorem A. ■■

5 Bases and rhombic tiling diagrams

In this section, we first define rhombic tiling diagrams (tilings) and their dual objects, wiring diagrams. Then we explain a correspondence between tilings and bases for

TP-functions on a box that are derived from the standard basis by a series of special mutations, so-called normal bases. Then we characterize the subsets of a box that can be extended to a normal basis. Finally, we demonstrate some applications of this characterization, in particular, that a TP-function on a box is extendable to a larger box.

5.1 Tilings and wirings

As before, let a be an n-tuple of positive integers. By a *rhombic tiling diagram*, or an *RT-diagram*, we mean the following construction.

In the half-plane $\mathbb{R} \times \mathbb{R}_+$ take n vectors $\xi_1, \ldots, \xi_n$ so that: (i) these vectors have equal Euclidean norms and are ordered clockwise around $(0,0)$, and (ii) all integer combinations of these vectors are different. Then the set $Z(a) := \{\lambda_1\xi_1 + \ldots + \lambda_n\xi_n \colon \lambda_i \in \mathbb{R},\ 0 \le \lambda_i \le a_i,\ i = 1, \ldots, n\}$ is a $2n$-gone. Moreover, $Z = Z(a)$ is a *zonogon*, as it is the sum of the segments $[0, a_i\xi_i]$. (Also it is a linear projection of the solid box $\mathrm{conv}(B(a))$ into the plane.) Its left boundary L, from the minimal point $p_0 := (0,0)$ to the maximal point $p_n := a_1\xi_1 + \ldots + a_n\xi_n$, is formed by the points (vertices) $p_i := a_1\xi_1 + \ldots + a_i\xi_i$ $(i = 0, \ldots, n)$ connected by the segments $p_{i-1}p_i := \{\lambda p_{i-1} + (1-\lambda)p_i \colon 0 \le \lambda \le 1\}$. Its right boundary R, from $p_0 =: p'_n$ to $p_n =: p'_0$, is formed by the points $p'_i := a_i\xi_i + \ldots + a_n\xi_n$ $(i = 0, \ldots, n)$ connected by the segments $p'_ip'_{i-1}$.

An *RT-diagram* D is a subdivision of the zonogon Z into "little" rhombi of the form $x + \{\lambda_i\xi_i + \lambda_j\xi_j \colon 0 \le \lambda_i, \lambda_j \le 1\}$ for some $i < j$ and a point x in Z. Such a rhombus is called an *ij-rhombus*. The diagram D may also be regarded as a directed planar graph $(V(D), E(D))$ whose vertices and edges are the vertices and side segments of the rhombi, respectively. Then each edge e corresponds to a parallel translation of some vector ξ_i and is directed accordingly. Two instances are illustrated in Fig. 1.

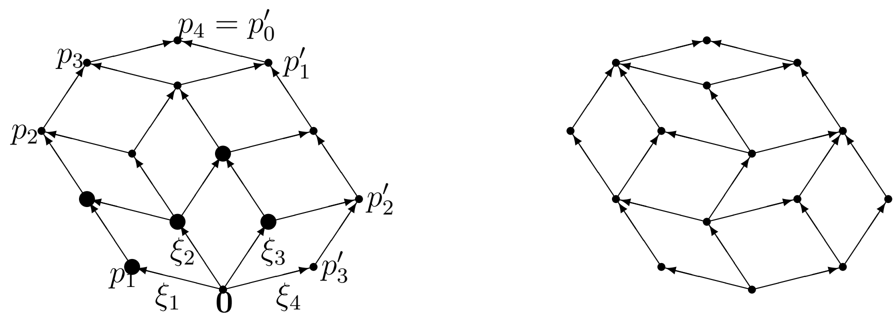


Figure 1: Instances of RT-diagrams for $n = 4$ and $a = (1, 2, 1, 1)$.

All diagrams (considered as graphs) are subgraphs of the graph G whose vertex set $V(G)$ consists of the points $\pi(x) := x_1\xi_1 + \ldots + x_n\xi_n$ for $x \in B(a)$ and whose edge set $E(G)$ is formed by the pairs $e = (\pi(x), \pi(x'))$ such that $\pi(x') = \pi(x) + \xi_i$ for some $i \in [n]$; we say that e is an *i-edge*. (Due to condition (ii) above, all points $\pi(x)$ are different; also $\pi(x)$ lies in Z.) Observe that G is graded in each color, in the

sense that for all $i \in [n]$ and $u, v \in V(G)$, any paths from u to v (when exist) have the same number of i-edges. The *height* $h(v)$ of a vertex v is the length (number of edges) of a path from the minimal vertex p_0 to v.

Let D be an RT-diagram. By a *dual path* in D we mean a maximal sequence $Q = (e_0, \rho_1, e_1, \dots, \rho_d, e_d)$, where: $\rho_1, \dots, \rho_d$ are rhombi of D, consecutive rhombi are different, and e_{k-1}, e_k are opposite edges in (the boundary of) ρ_k, $k = 1, \dots, d$. Then all edges in Q are i-edges for some $i \in [n]$, and Q connects the left boundary L and the right boundary R of the zonogon Z. We say that Q is a *dual i-path* and assume that Q is oriented so that its first edge e_0 belongs to L (and e_d belongs to R). So, for each $i \in [n]$, there are exactly a_i dual i-paths and these paths are pairwise disjoint. In particular, if the first edge e_0 of a dual i-path Q is a qth edge in the side $p_{i-1}p_i$ of Z (i.e., it connects the point $p_{i-1} + (q-1)\xi_i$ and $p_{i-1} + q\xi_i$ for some $q \in \{1, \dots, a_i\}$), then the last edge e_d of Q is the qth edge in the side $p'_i p'_{i-1}$; we denote such edges in L and R by $\ell_{i,q}$ and $r_{i,q}$, respectively. Also one can see that a dual i-path intersects each dual j-path with $j \neq i$ at exactly one rhombus.

In light of these observations and using planar duality, the RT-diagrams can be associated with so-called wiring diagrams, or W-diagrams (giving the de Bruijn dualization [2]; we use the shorter name "wire" rather than "de Bruijn line"). A *W-diagram* is represented by $|a|$ curves, or *wires*, $\sigma_{i,q}$ for $i = 1, \dots, n$ and $q = 1, \dots, a_i$, such that

(33) each $\sigma_{i,q}$ is identified with (the image of) a continuous injective map σ of the segment $[0,1]$ into Z such that $\sigma(0) = p_{i,q}$, $\sigma(1) = p'_{i,q}$, and $\sigma(t)$ lies in the interior of Z for $0 < t < 1$, where $p_{i,q}$ and $p'_{i,q}$ are the mid-points of the boundary edges $\ell_{i,q}$ and $r_{i,q}$, respectively, and the following conditions hold:

- (a) any two wires $\sigma_{i,q}$ and $\sigma_{i,q'}$ ($q \neq q'$) are disjoint, i.e., there are no $0 \leq t, t' \leq 1$ such that $\sigma_{i,q}(t) = \sigma_{i,q'}(t')$;
- (b) any two wires $\sigma_{i,q}$ and $\sigma_{j,q'}$ with $i \neq j$ have exactly one point in common, i.e., there are unique t, t' such that $\sigma_{i,q}(t) = \sigma_{j,q'}(t')$ (these wires *cross*, not touch, at this point);
- (c) no three wires have a common point.

Such a diagram is considered up to a homeomorphism of the zonogon Z stable on its boundary, and when needed, we may assume that each wire is piece-wise linear. Also we orient each wire $\sigma_{i,q}$ from $p_{i,q}$ to $p'_{i,q}$ and call it an *i-wire*.

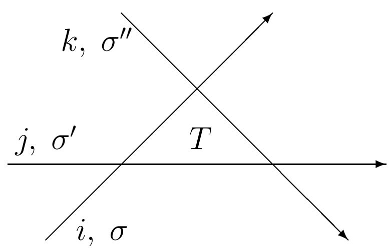
There is a natural one-to-one correspondence between the RT- and W-diagrams. More precisely, for an RT-diagram D and a dual path $Q = (\ell_{i,q} = e_0, \rho_1, e_1, \dots, \rho_d, e_d = r_{i,q})$ in it, take as wire $\sigma_{i,q}$ the concatenation of the segments connecting the mid-points of edges e_k, e_{k+1}; then these wires form a W-diagram. (We will call such a $\sigma_{i,q}$ the *median line* of Q.) Conversely, given a W-diagram W, consider the common point v of wires $\sigma_{i,q}$ and $\sigma_{j,q'}$ with $i < j$ in W. For $k = 1, \dots, n$, let x_k be the number of wires $\sigma = \sigma_{k,q''}$ that *go below* v, i.e., v and p_n occur in the same connected region when the curve σ is removed from Z. Then we associate to v the ij-rhombus with the minimal vertex at the point $x_1\xi_1 + \ldots + x_n\xi_n$. One can

check that conditions (a)–(c) in (33) provide that the set of rhombi obtained this way forms an RT-diagram D and, furthermore, that the W-diagram constructed from D as explained above is equivalent to W.

Depending on the context, a W-diagram W may also be regarded as a directed planar graph $(V(W), E(W))$, which is a sort of dual graph of the corresponding RT-diagram D. More precisely, $V(W)$ consists of the intersection points of pairs of wires (corresponding to the rhombi of D) plus the points $p_{i,q}$ and $p'_{i,q}$ for all i, q, and $E(W)$ consists of the wire parts between such vertices, with the orientation inherited from that of the wires. An edge contained in an i-wire is called an *i-edge*. Observe that:

(34) for the intersection point v of an i-wire and a j-wire with $i < j$, the edges e_i, e_j, e'_i, e'_j incident with v follow in this order clockwise around v, where e_i, e'_i are the i-edges entering and leaving v, respectively, and e_j, e'_j are the j-edges entering and leaving v, respectively.

The assertions in the rest of this section will be stated in terms of RT-diagrams. In its turn, the language of W-diagrams will mainly be used to simplify visualization and technical details in the proofs. In particular, we will sometimes choose triples of wires in a W-diagram, an i-wire σ, a j-wire σ' and a k-wire σ'' for $i < j < k$, and consider the region (curvilinear *triangle*) $T = T(\sigma, \sigma', \sigma'')$ in Z bounded by the parts of these wires between their intersection points. We will distinguish between two cases, by saying that $\sigma, \sigma', \sigma''$ form the *Δ-configuration* if the point $\sigma \cap \sigma''$ (as well as the triangle T) lies above σ', and the *∇-configuration* if this point lies below σ'; see Fig. 2.

(a) 

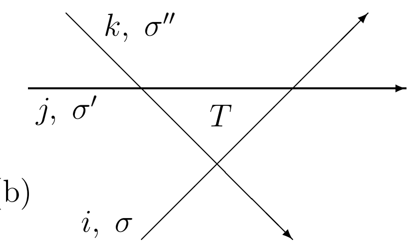
(b) 

Figure 2: (a) Δ-configuration; (b) ∇-configuration.

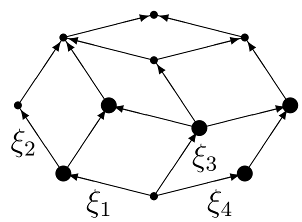
5.2 RT-diagrams and bases

In this subsection we discuss relationships between RT-diagrams and TP-bases in a box $B(a)$. For $S \subseteq B(a)$, let G_S denote the induced subgraph of G with the vertex set $\pi(S)$ (the graph G is defined in the previous subsection). We are interested in those subsets S for which G_S is an RT-diagram, in which case we say that S is an *RT-set*.

An important instance of RT-sets is the set $Int(a)$ of fuzzy-intervals (including the zero vector), i.e., the standard basis for $B(a)$; see the left part in Fig. 1 where the RT-diagram for $Int(1,2,1,1)$ is drawn. (This fact can be shown by induction

on n. Instruction: in case $n=2$, $G_{Int(a)}$ is the direct product $L_{a_1}\times L_{a_2}$, where L_q is the path of length q; so it is an RT-diagram. In case $n>2$, let $a'=(a_1,\dots,a_{n-1})$ and assume by induction that $G'=G_{Int(a')}$ is an RT-diagram. Let R' be the right boundary of G' (going from the minimal vertex $(0,0)$ to the maximal vertex $a_1\xi_1+\ldots+a_{n-1}\xi_{n-1}$). Then $G_{Int(a)}$ is obtained by taking the corresponding union of G' and $R'\times L_{a_n}$.)

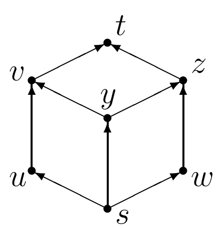
Let $\mathcal{M}(a)$ be the set of bases which can be obtained from $Int(a)$ by a series of TP3-mutations. At the first glance, it seems likely that each member of $\mathcal{M}(a)$ is an RT-set. However, this is not so. Indeed, in the standard basis for $B(1,2,1,1)$, take the elements $1_1, 1_1+1_2, 1_2, 1_3, 1_2+1_3$; their images are indicated by thick dots on the left picture in Fig. 1. These elements give rise to the TP3-mutation $1_2\rightsquigarrow 1_1+1_3$ of $Int(a)$, but the resulting basis is already not an RT-set. (By withdrawing the fourth coordinate, we obtain a similar situation in the 3-dimensional box $B(1,2,1)$.) Another example, with the Boolean cube $2^{[4]}$, is drawn in the picture below (here the RT-diagram determines a basis $\mathcal{B}$ but the mutation involving the sets corresponding to the thick dots, namely, $\{3\}\rightsquigarrow\{1,4\}$, results in a basis which is not an RT-set).



So, to get bases corresponding to RT-diagrams, we have to restrict the class of mutations that we apply. Consider a basis $\mathcal{B}$ and a cortege (x,i,j,k) such that the vectors involving in (3), except for one vector $y\in\{x+1_j, x+1_i+1_k\}$, belong to $\mathcal{B}$. We say that the mutation $y\rightsquigarrow y'$ (where $\{y,y'\}=\{x+1_j,x+1_i+1_k\}$) is *normal* if both vectors x and $x+1_i+1_j+1_k$ belong to $\mathcal{B}$ as well. In this case, for $s:=\pi(x)$, the six points

$$s,\quad u:=s+\xi_i,\quad v:=s+\xi_i+\xi_j,\quad w:=s+\xi_k,\quad z:=s+\xi_j+\xi_k,\quad t:=s+\xi_i+\xi_j+\xi_k, \tag{35}$$

along with the six edges connecting them, form a (little) *hexagon* in $G_{\mathcal{B}}$, denoted by $H(s,i,j,k)$. This hexagon H is partitioned into three rhombi in D, either by use of edges sy, yv, yz or by use of edges uy, wy, yt, where y is the unique vertex of D in the interior of H; we refer to H as a $\curlyvee$-*hexagon* in the former case, and a $\curlywedge$-*hexagon* in the latter case, see Fig. 3.

(a) 

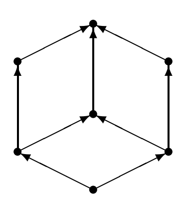
(b) 

Figure 3: (a) $\curlyvee$-hexagon; (b) $\curlywedge$-hexagon.

It is useful to describe the difference between $\curlyvee$- and $\curlywedge$-hexagons of D in terms of the corresponding wires in the W-diagram W related to D. More precisely, given a hexagon $H = H(s,i,j,k)$, let σ,σ',σ'' be the wires in W corresponding to the dual i-, j- and k-paths, respectively, that meet rhombi in H. It is easy to see that these wires form the Δ-configuration if H is a $\curlyvee$-hexagon, and the ∇-configuration if H is a $\curlywedge$-hexagon (cf. Figures 2 and 3). Moreover, in both cases the triangle $T(\sigma,\sigma',\sigma'')$ is *inseparable*, which means that none of the other wires in W goes across this triangle. A converse property also takes place: if the triangle for three wires σ,σ',σ'' (concerning different $i,j,k\in[n]$) is inseparable, then the three rhombi in D corresponding to the points where these wires intersect are assembled in a hexagon.

Let us call a basis for $B(a)$ *normal* if it can be obtained from the standard basis $Int(a)$ by a series of normal TP3-mutations. We show the following

Proposition 5.1 *Each normal basis induces an RT-diagram, and vice versa.*

Proof. Let $\mathcal{B}$ be a basis such that $D := G_{\mathcal{B}}$ is an RT-diagram. Then a normal mutation in $\mathcal{B}$ corresponds to a transformation within some hexagon H of D. This transformation changes the partition of H into three rhombi by the other partition of H into three rhombi, and we again obtain an RT-diagram. This implies that each normal basis is an RT-set (since the standard basis is such).

Next we prove the other direction in the proposition: each RT-diagram D corresponds to a normal basis.

To show this, we use induction on the parameter $h(D) := \sum_{v\in V(D)} h(v)$, the (total) *height of* D. Suppose D contains a $\curlywedge$-hexagon H. Then the transformation of H into a $\curlyvee$-hexagon (which matches a normal mutation) results in an RT-diagram D' such that $h(D') = h(D) - 1$ (as the distance from the minimal vertex of H to the vertex of the diagram occurring in the interior of H changes from 2 to 1). By induction, assuming that D' corresponds to a normal basis, a similar property takes place for D'.

To complete the induction, it remains to consider the situation when no $\curlywedge$-hexagon exists. We assert that

(36) if D contains no $\curlywedge$-hexagon, then D corresponds to the standard basis.

To show this, consider the W-diagram W related to D and take the wire τ in W that goes from the point p_{n,a_n} to p'_{n,a_n}. Let Ω be the region in Z between τ and the right boundary R. Two cases are possible.

Case 1: No two wires in W intersect within the interior of Ω. This means that the dual n-path Q in D beginning at the last edge ℓ_{n,a_n} of L and ending at the edge r_{n,a_n} *follow the right boundary* R of Z. Then we can reduce the diagram D by removing R and (the interiors of the elements of) Q. This gives a correct RT-diagram for the tuple $a' = (a_1,\ldots,a_{n-1},a_n-1)$, and (36) follows by applying induction on $|a|$. (When $a_n = 1$, the nth entry of a' vanishes.)

Case 2: There are two wires in W that intersect in the interior of Ω, say, an i'-wire μ and a j'-wire ν. Then i',j',n are different. Let for definiteness $i' < j'$.

Then the end point of μ occurs in R later than the end point of ν. This implies that the point $\mu \cup \tau$ lies below ν, i.e., μ, ν, τ form the ∇-configuration.

So W contains three wires $\sigma, \sigma', \sigma''$ forming the ∇-configuration. Let $\eta(\sigma, \sigma', \sigma'')$ denote the number of wires $\overline{\sigma} \in W$ such that (the interior of) the triangle $T = T(\sigma, \sigma', \sigma'')$ lies entirely below $\overline{\sigma}$. Choose such wires $\sigma, \sigma', \sigma''$ so that $\eta(\sigma, \sigma', \sigma'')$ is maximum and, subject to this, the area of their triangle T is as small as possible. Let for definiteness $\sigma, \sigma', \sigma''$ be, respectively, i-, j- and k-wires with $i < j < k$, and let I, J, K denote the i-, j- and k-sides of T. (In view of (34), J, K are directed from the point $\sigma' \cap \sigma''$, and I is directed from $\sigma \cap \sigma''$.)

Claim. *The triangle T is inseparable.*

Proof of the Claim. Suppose there exists a p-wire $\widehat{\sigma}$ going across T. Then $\widehat{\sigma}$ crosses two sides among I, J, K. We consider five possible cases and use property (34).

(a) $\widehat{\sigma}$ first crosses K and then crosses J. Then $p < j < k$. Form the triple $\Sigma := \{\widehat{\sigma}, \sigma', \sigma''\}$.

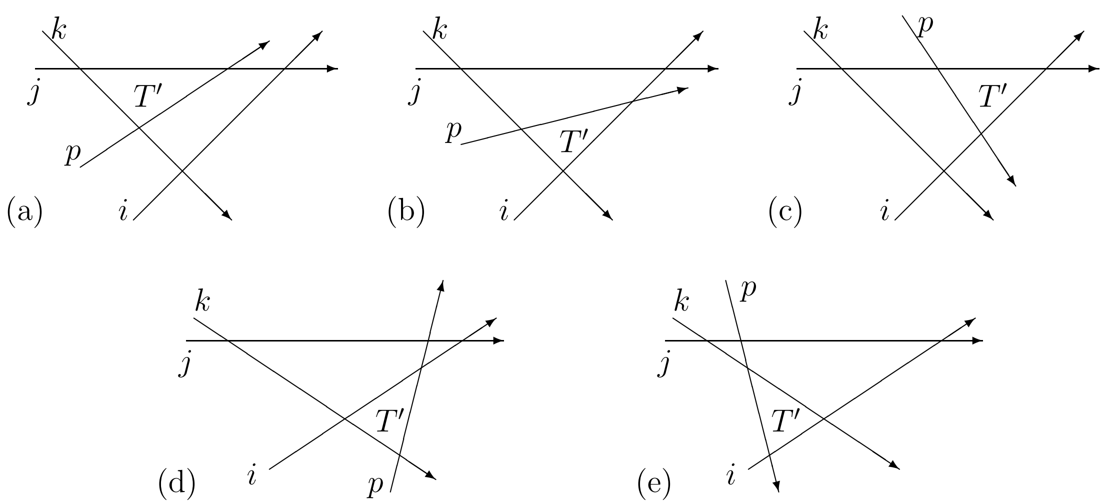
(b) $\widehat{\sigma}$ first crosses K and then crosses I. Then $i < p < k$. Form $\Sigma := \{\sigma, \widehat{\sigma}, \sigma''\}$.

(c) $\widehat{\sigma}$ first crosses J and then crosses I. Then $i < j < p$. Form $\Sigma := \{\sigma, \sigma', \widehat{\sigma}\}$.

(d) $\widehat{\sigma}$ first crosses I and then crosses J. Then $p < i < k$. Form $\Sigma := \{\widehat{\sigma}, \sigma, \sigma''\}$.

(e) $\widehat{\sigma}$ first crosses J and then crosses K. Then $i < k < p$. Form $\Sigma := \{\sigma, \sigma'', \widehat{\sigma}\}$.

(The case when $\widehat{\sigma}$ first crosses I and then crosses K is impossible, otherwise we would have $p < i$ and $k < p$, by (34).) Let T' denote the triangle of the triple Σ. Cases (a)–(e) are illustrated in the picture.



In all cases, Σ forms the ∇-configuration. In cases (a)–(c), the triangle T' lies inside T, which implies that $\eta(\Sigma) \geq \eta(\sigma, \sigma', \sigma'')$ and that the area of T' is smaller than that of T. In cases (d),(e), it is easy to see that when T lies below some wire, then so does T'. Also, in case (d) (resp. (e)), T' lies below σ (resp. σ''), whereas T does not. Therefore, in these two cases, $\eta(\Sigma) > \eta(\sigma, \sigma', \sigma'')$. So we come to a contradiction with the choice of $\sigma, \sigma', \sigma''$, yielding the Claim. $\blacksquare$

By the Claim, the three rhombi corresponding to the vertices of T form a $\curlywedge$-hexagon in D. This contradiction completes the proof of (36) and of the proposition. ■

Remark 3. As is seen from the above proof, the diagram D corresponding to the standard basis $Int(a)$ is *minimal* in the sense that the height $h(D)$ of vertices of D is minimum among all RT-diagrams. Applying the central symmetry to D and reversing all the edges, we obtain the RT-diagram D^* having the maximum height. The basis corresponding to D^* is formed by the "complementary" vectors $a-x$ of the fints x (*co-fints*). By Proposition 5.1, this "complementary basis" $Int^*(a)$ is normal as well. For a similar reason, any normal basis $\mathcal{B}$ and its complementary basis $\mathcal{B}^* := \{x : a-x \in \mathcal{B}\}$ are connected by a series of normal mutations.

It is not clear to us whether, for some tuple a, there exists a TP-basis beyond $\mathcal{M}(a)$.

5.3 Normal bases including a given set.

An interesting open question is to characterize the subsets of $B(a)$ that can be extended to a TP-basis. In this subsection we consider a different but somewhat related problem:

(37) Given a set $X \subset B(a)$, decide whether there exists a normal basis for $B(a)$ including X, or, equivalently, an RT-diagram D whose vertex set contains all points of $\pi(X)$ (the image of X in the zonogon $Z(a)$).

As a variant of such a problem (in fact, equivalent to (37)), one is given a set of (not necessarily disjoint) subzonogons in $Z(a)$, with possible rhombic tilings on some of them, and is asked of their extendability to a tiling on $Z(a)$.

It is useful to reformulate (37) in terms of wiring diagrams. Then each point $x=(x_1,\ldots,x_n) \in X$ imposes a requirement on the set of wires *going below* (the image of) x in a W-diagram $W=\{\sigma_{i,q}\}$, assuming that the desired RT-diagram does exist and regarding the wires in W as being realized by the median lines of the dual paths. More precisely, if (37) has a solution, then

(38) for each $i=1,\ldots,n$, the wires $\sigma_{i,1},\ldots,\sigma_{i,x_i}$ should go below x, while the other i-wires above x.

Consider three wires $\sigma=\sigma_{i,q}$, $\sigma'=\sigma_{j,q'}$ and $\sigma''=\sigma_{k,q''}$ with $i<j<k$. Suppose there is a point $x \in X$ such that $x_i<q$, $x_j \ge q'$ and $x_k<q''$. Then (38) forces σ,σ',σ'' to form the Δ-configuration. Another possible situation is when $x_i \ge q$, $x_j<q'$ and $x_k \ge q''$. In this case, by similar reasons, the point x prescribes the ∇-configuration for σ,σ',σ''.

These observations lead us to the following conclusion: problem (37) has no solution if

(39) there are $1 \le i<j<k \le n$ and two points $x,x' \in X$ such that $x_i<x'_i$, $x_j>x'_j$ and $x_k<x'_k$.

The simplest example is given by the points 2 and 13 of the Boolean cube $2^{[3]}$, which cannot simultaneously occur in one and the same RT-diagram.

It turns out that (39) fully describes obstacles to the solvability of the problem.

Theorem 5.2 *Problem (37) has solution if and only if (39) does not take place.*

Proof. We have to show the solvability of (37) if no i, j, k, x, x' as in (39) exist (the other direction in the theorem has been explained). Our proof is constructive and provides a polynomial time algorithm of finding a required tiling.

Let $k \in [n]$ and $q \in [a_k]$ and suppose that the wires $\sigma_{i,p}$ are already constructed for all (i, p) such that either $i < k$, or $i = k$ and $p < q$. This means that the W-diagram W' formed by these wires is realized by use of a directed planar graph $H = (V(H), E(H))$ embedded in Z (according to (33)) and that the requirements as in (38) are satisfied; we add to H the boundary of Z as well. So $V(H)$ consists of the intersection points of wires plus the points $p_{j,r}$ and $p'_{j,r}$ for all $j \in [n]$ and $r \in [a_j]$. Let $F(H)$ be the set of inner faces of H. We also (conditionally) place each point $x \in X$ in the interior of some face $f \in F(H)$, which is chosen according to (38), i.e., for each $\sigma = \sigma_{i,p} \in W'$, x occurs below σ if $x_i \le p$, and above σ otherwise (so f is chosen uniquely for x).

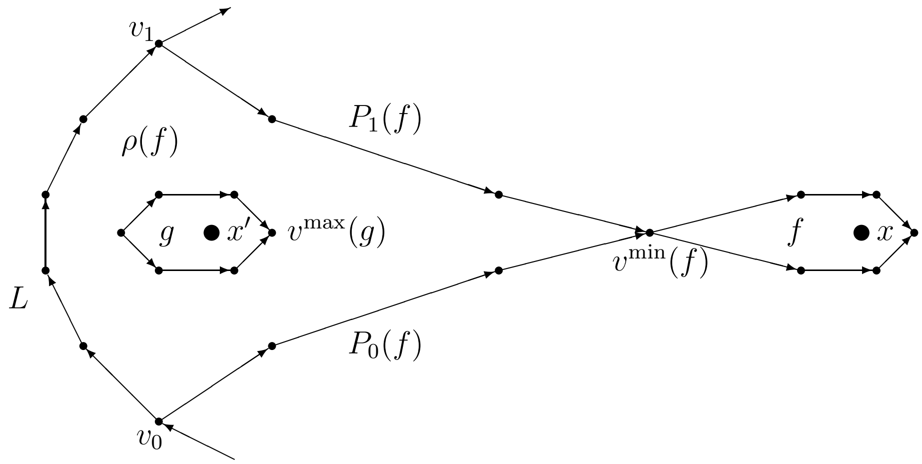
The faces in $F(H)$ containing the minimal vertex p_0 and the maximal vertex p_n of Z are denoted by $f^{\min}$ and $f^{\max}$, respectively. It follows from (34) that H is acyclic. When an edge e is an i-edge (a part of an i-wire), we also say that e has *index* i.

Our aim is to add wire $\sigma = \sigma_{k,q}$ to W' by transforming H in a due way. Let $X_\downarrow$ be the set of points of $x \in X$ with $x_k \le q$, and $X^\uparrow := X - X_\downarrow$ (the sets of points that should lie below and above σ, respectively).

Consider a face f. Its boundary ∂f has two vertices v, v' such that ∂f is formed by two (directed) paths beginning at v and ending at v'; we call v, v' the *minimal* and *maximal* vertices of f and denote them by $v^{\min}(f)$ and $v^{\max}(f)$, respectively. (When ∂f has no common edge with the boundary ∂Z of Z, this fact can be seen by considering the vertex w corresponding to the face f in the RT-diagram D' related to W'. Then one path in ∂f corresponds to the sequence (in the clockwise order) of the edges in D' leaving w, and the other path to the sequence of edges entering w. When ∂f meets an edge in ∂Z, reasonings are easy as well.)

Let f contain a point $x \in X_\downarrow$. The fact that H is acyclic implies that there is a path $P = P_0(f)$ in H such that: (a) P begins at a vertex v_0 of L, ends at $v^{\min}(f)$ and has all intermediate points not in L, and (b) for any edge $e = (u, v)$ of P and for the other edge e' of H entering v, the index of e is less than the index of e'. Such a path is constructed uniquely. Symmetrically, there is a path $P' = P_1(f)$ such that: (a') P' begins at a vertex v_1 of L, ends at $v^{\min}(f)$ and has all intermediate points not in L, and (b') for any edge $e = (u, v)$ of P' and for the other edge e' of H entering v, the index of e is greater than the index of e'.

Clearly v_0 occur in L earlier than v_1 (and one may say that $P_0(f)$ goes below $P_1(f)$). Let $\rho(f)$ denote the closed region bounded by these paths and the path in L from v_0 to v_1. See the picture for illustration.



Let us say that a closed region ρ in Z formed as the union of some faces and edges of H is an *ideal* one if each edge $e \not\subset \partial Z$ having an end vertex in ρ but the interior not in ρ is directed from ρ to $Z - \rho$. In view of (34),

(40) (i) the indices of edges are weakly increasing along the path $P_0(f)$ and weakly decreasing along $P_1(f)$; and (ii) $\rho(f)$ is ideal.

We assert that $\rho(f)$ contains no point from $X^{\uparrow}$. Suppose this is not so and such a point x' exists. Let x' lie in a face g. We are going to show the existence in W' of an i-wire σ and a j-wire σ' with $i < j$ such that

(41) f lies below σ and above σ', while g lies above σ and below σ'.

To show this, consider the edges e, e' entering $v^{\max}(g)$, and assume that e is an i'-edge, e' is a j'-edge, and $i' < j'$. Take the i'-wire τ containing e; then g lies above τ. If f lies below τ, then we take τ as the desired σ in (41). Now suppose that f lies above τ. Using (34) and (40)(i), one can conclude that τ meets the path $P_0(f)$, but not $P_1(f)$, at some vertex v. Let e be the edge in $P_0(f)$ beginning at v and take the wire σ containing e; let it be an i-wire. Then (34) and (40)(i) imply that σ meets $P_1(f)$, whence the face f lies below σ. Also $i < i'$, implying that the face g lies above σ. A j-wire σ' required in (41) is constructed in a similar way, starting with the edge e' and using, if needed, the path $P_1(f)$. By the construction, we have $i < j$.

Since f lies below σ and g lies above σ, we have $x_i < x'_i$. In its turn, the behavior of σ' gives $x_j > x'_j$. Also $x_k < q \le x'_k$. Note that $j < k$; for if $j = k$ then σ' would be of the form $\sigma_{k,q'}$ with $q' < q$, giving $x'_k < q' < q \le x'_k$. So $i < j < k$, and we obtain (39); a contradiction.

Thus, the region $\rho(f)$ contains no elements of $X^{\uparrow}$. Define

$$\widehat{\rho} := \cup\{\rho(f) \colon f \in F(H) \;\; f \text{ contains an element of } X_{\downarrow}\}.$$

Then $\widehat{\rho} \cap X^{\uparrow} = \emptyset$. By the construction of regions $\rho(f)$ and by (40)(ii), $\widehat{\rho} \cap \partial Z$ is contained in the part of L from $p_{1,1}$ to the vertex in $L \cap V(H)$ preceding $p_{k,q}$. Also

(42) (i) for each face f with $f \cap X_{\downarrow} \neq \emptyset$, the vertex $v^{\min}(f)$ is in $\widehat{\rho}$; and (ii) $\widehat{\rho}$ is ideal.

Now we are ready to draw the desired wire $\sigma_{k,q}$. We add to $\widehat{\rho}$ the part of L from p_0 to $p_{k,q}$. Obviously, $p_{k,q}$ belongs to the upmost face $f^{\max}$. Let U be the set of edges $e \not\subset \partial Z$ connecting $\widehat{\rho}$ and $Z - \widehat{\rho}$; they go out of $\widehat{\rho}$, by (42)(ii). Let $\mathcal{F}$ be the set of faces f such that $v^{\min}(f) \in \widehat{\rho}$ but $f \not\subseteq \widehat{\rho}$.

First consider the case $q = 1$. Then $p'_{k,q}$ belongs to the bottommost face $f^{\min}$. For each edge $e \in U$, choose a point v_e in the interior of e. Clearly each face $f \in \mathcal{F} - \{f^{\min}, f^{\max}\}$ has exactly two edges e, e' in U. We connect v_e and $v_{e'}$ by a curve $\sigma(f)$ within f, subdividing f into two regions (faces). If f has points from $X_{\downarrow}$ ($X^{\uparrow}$), we move them into the region containing $v^{\min}(f)$ (resp. $v^{\max}(f)$). We act similarly for the face $f = f^{\max}$ ($f = f^{\min}$) with the only difference that $\sigma(f)$ connects $p_{k,q}$ (resp. $p'_{k,q}$) and v_e for the unique edge $e \in U$ in f. The concatenation of these $\sigma(f)$ for all $f \in \mathcal{F}$ gives the desired wire $\sigma_{k,q}$ (it intersects each wire in W' exactly once, by (42)(ii)).

In case $q > 1$, we extend $\widehat{\rho}$ by adding to it the region ρ' of Z formed by the faces lying below $\sigma_{k,q-1}$. Note that no point of $X^{\uparrow}$ is contained in ρ' and that no edge of H goes from $Z - \rho'$ to ρ'. Then (42) remains valid, and we draw $\sigma_{k,q}$ as in the previous case.

Add $\sigma_{k,q}$ to W' and continue the process. Upon constructing the last wire σ_{n,a_n}, the resulting W-diagram W determines a required RT-diagram for X. (Observe that if a face f for W involves a point of X, then such a point x is unique and the vertex of D corresponding to f is just $\pi(x)$.) ∎

Remark 3. Analyzing the above proof, one can deduce that the height $h(D)$ of the obtained RT-diagram D is minimum among the possible tilings for X. Moreover, such a D is unique.

5.4 Sub-zonogons and sub-boxes.

We demonstrate one simple consequence of Theorem 5.2 that will be used later.

Let $y, a' \in \mathbb{Z}^n_+$ be such that $y + a' \leq a$. Then the *sub-zonogon* of $Z(a)$ of size a' with the beginning at y is the set $Z(y; a') := y + Z(a')$ (clearly it is contained in $Z(a)$). In other words, $Z(y; a')$ is the projection by π of the convex hull of the sub-box $B(y|\, y + a') := \{x \in \mathbb{Z}^n_+ \colon y \leq x \leq y + a'\}$ of the box $B(a)$ (note that we admit $a_i = 0$ for some i's, i.e., the dimension of the sub-box may be less than n).

Proposition 5.3 *Any RT-diagram D' for a sub-zonogon $Z(y; a')$ is extendable into an RT-diagram for $Z(a)$.*

Proof. It suffices to consider the set X of points $x \in B(y|\, y + a')$ such that $\pi(x)$ lies in the boundary of $Z(y; a')$ and show that (37) has a solution for this X. By Theorem 5.2, one has to check that (39) does not take place. This is straightforward, taking into account that for the elements $x \in X$ are coordinate-wise weakly

increasing when $\pi(x)$ moves along the the left boundary of $Z(y;a)$, and decreasing along the right boundary. ■

(An alternative proof: Assuming that $B' := B(y|\,y+a')$ is not the whole $B(a)$, there is an i such that the sub-box B'' of the form either $B(\mathbf{0}|\,a-1_i)$ or $B(1_i|\,a)$ includes B'. By induction on the size of the box, one may assume that D' is extendable into an RT-diagram D'' on the sub-zonogon Z'' in $Z(a)$ corresponding to B''. To extend D'' into an RT-diagram on $Z(a)$ is easy.)

Next, as is explained in the Introduction, any TP-function on a truncated box can be extended into a TP-function on the entire box. Using Proposition 5.3, we can further extend the latter function.

Proposition 5.4 *Let $B' = B(a'|\,a'')$ be a sub-box of a box $B(a)$ and let f' be a TP-function on B'. Then f' is extendable into a TP-function f on $B(a)$.*

Proof. Take a normal basis $\mathcal{B}'$ for B', e.g., the standard one, and let g' be the restriction of f' to $\mathcal{B}'$. Extend the RT-diagram D' corresponding to $\mathcal{B}'$ into an RT-diagram on $Z(a)$. This gives a basis $\mathcal{B}$ for $B(a)$ including $\mathcal{B}'$. Extend g' arbitrarily into a function g on $\mathcal{B}$. Then the TP-function f on $B(a)$ determined by g is as required. ■

Remark 4. In a recent work, Henriques and Speyer [11] present a number of results on rhombus (viz. rhombic) tilings and their applications to the n-cube recurrence, and others. They argue in direct terms of tilings, not appealing to wiring diagrams. On this way, they prove Proposition 5.3 for the case when $Z(y;a')$ is a hexagon. Also it is shown there that each rhombus tiling can be turned into the minimal rhombus one by a series of downward flips (viz. transformations of $\curlywedge$-hexagons into $\curlyvee$-ones), yielding an alternative proof of Proposition 5.1. They also consider a certain complex associated with the set of tilings on a zonogon, prove that it is simply connected, and use this fact to show that the vertices of tiling index a basis for the functions obeying the n-cube recurrence. We, however, do not see whether one can apply a similar approach to the case of TP-functions (since not every TP3-relation concerns a hexagon of a tiling, as we have seen above).

6 Submodular TP-functions

In this section we consider TP-functions on a box $B(a)$ with the additional property of submodularity. We demonstrate an important role of the standard basis $Int(a)$ for such functions by showing that a TP-function is submodular if and only if its restriction to $Int(a)$ is such. Here $Int(a)$ is the set of all fuzzy-intervals in $B(a)$ to which the zero fuzzy-interval is added as well.

Recall that a function f on a lattice $\mathcal{L}$, with meet operation $\wedge$ and join operation $\vee$, is called *submodular* if it satisfies the *submodular inequality*

$$f(\alpha)+f(\beta)\ge f(\alpha\wedge\beta)+f(\alpha\vee\beta)$$

for each pair $\alpha, \beta \in \mathcal{L}$. Sometimes one considers a function f on a part $\mathcal{L}'$ of the lattice, in which case the submodular inequality is imposed whenever all $\alpha, \beta, \alpha \vee \beta, \alpha \wedge \beta$ occur is $\mathcal{L}'$.

The lattice operations on elements x, x' of the box $B(a)$ are defined in a natural way: $x \wedge x'$ and $x \vee x'$ are the vectors whose ith entries are $\min\{x_i, x'_i\}$ and $\max\{x_i, x'_i\}$, respectively. A simple fact is that a function f on the lattice $B(a)$ is submodular if and only if

$$f(x + 1_i) + f(x + 1_j) \geq f(x) + f(x + 1_i + 1_j) \tag{43}$$

holds for all x, i, j $(i \neq j)$ such that all four vectors involved belong to $B(a)$.

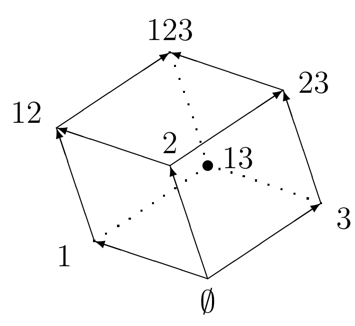
Theorem 6.1 *Let f be a TP-function on a box $B(a)$. Then f is submodular if and only if it is submodular on the standard basis $Int(a)$, where the latter means that (43) holds whenever $i \neq j$ and the four vectors occurring in it belong to $Int(a)$.*

Proof. We use results on rhombic tilings from Section 5.

Consider elements $x, x + 1_i, x + 1_j, x + 1_i + 1_j$ of $B(a)$ $(i \neq j)$. Their images in the zonogon $Z(a)$ form a (little) rhombus, and by (a very special case of) Proposition 5.3, this rhombus belongs to some RT-diagram on $Z(a)$. In other words, the above four elements are contained in some normal basis for $B(a)$. In light of this, we can reformulate the theorem (and thereby slightly strengthen it) by asserting that if a TP-function f is submodular with respect to some normal basis $\mathcal{B}$ (or its corresponding tiling), then f is submodular w.r.t. any other normal basis. (When saying that f is *submodular w.r.t.* $\mathcal{B}$, we mean that (43) holds whenever the four vectors there belong to $\mathcal{B}$. The theorem considers as $\mathcal{B}$ the standard basis $Int(a)$.)

Next, we know (see the proof of Proposition 5.1) that making normal mutations (equivalently: transformations of $\curlywedge$-hexagons into $\curlyvee$-hexagons, or conversely), one can reach any normal basis from a fixed one. Therefore, it suffices to show that the submodularity is maintained by a normal mutation.

In other words, it suffices to prove the theorem for the simplest case when $B(a)$ is the 3-dimensional Boolean cube $C = 2^{[3]}$. In this case, the standard basis Int consists of the sets $\emptyset, 1, 2, 3, 12, 23, 123$, the submodularity on Int involves the three rhombi of the corresponding tiling, and one has to check the submodularity for the three rhombi arising under the mutation $2 \rightsquigarrow 13$; see the picture.



Let f be a TP-function on C, i.e., f satisfies

$$f(2) + f(13) = \max\{f(1) + f(23), f(3) + f(12)\}. \tag{44}$$

The submodularity on Int reads as:

$$
\begin{aligned}
f(\emptyset) + f(23) &\leq f(2) + f(3); && (45)\\
f(\emptyset) + f(12) &\leq f(1) + f(2); && (46)\\
f(2) + f(123) &\leq f(12) + f(23). && (47)
\end{aligned}
$$

We show that (44)–(47) imply the submodular inequalities for the other three rhombi, as follows. Adding $f(1)$ to (both sides of) (45) gives

$$f(1) + f(23) \leq f(2) + f(3) - f(\emptyset) + f(1).$$

Adding $f(3)$ to (46) gives

$$f(3) + f(13) \leq f(1) + f(2) - f(\emptyset) + f(3).$$

Substituting these inequalities into (44), we obtain

$$f(2) + f(13) = \max(f(1) + f(23), f(3) + f(12)) \leq f(1) + f(2) + f(3) - f(\emptyset),$$

which implies the submodular inequality for the rhombus on $\emptyset, 1, 3, 13$:

$$f(\emptyset) + f(13) \leq f(1) + f(3).$$

Arguing similarly, one obtains the submodular inequalities for the rhombi on $1, 12, 13, 123$ and on $3, 13, 23, 123$. More precisely:

$$
\begin{aligned}
f(1) + f(123) &\leq f(1) + f(12) + f(23) - f(2) && \text{(by (47))}\\
&\leq f(2) + f(13) + f(12) - f(2) && \text{(by (44))}\\
&= f(13) + f(12);
\end{aligned}
$$

and

$$
\begin{aligned}
f(3) + f(123) &\leq f(3) + f(12) + f(23) - f(2) && \text{(by (47))}\\
&\leq f(2) + f(13) + f(23) - f(2) && \text{(by (44))}\\
&= f(13) + f(23).
\end{aligned}
$$

(Note that if needed, one can reverse the arguments to obtain (45)–(47) from the other three inequalities.) ■

Remark 5. If we replace in Theorem 6.1 the submodularity condition by the corresponding *supermodularity* condition (i.e., replace $\geq$ by $\leq$), then the TP-function f need not be supermodular globally, even in the Boolean case with $n = 3$. A counterexample is the function on $2^{[3]}$ taking value 0 on $\{\emptyset\}, 1, 2, 3, 12$ and value 1 on $13, 23, 123$ (the supermodularity is violated for the sets 13 and 23). On the other hand, one can show that a version of the theorem concerning *modular* TP-functions is valid.

7 Skew-submodular TP-functions

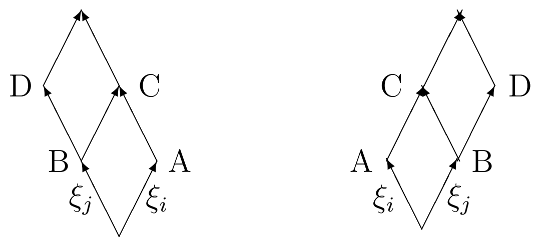
In this section we show that another important property can also be TP-propagated from the standard basis to the entire box.

Definition. We say that a function f on a box $B(a)$ is *skew-submodular* if

$$f(x+1_i+1_j)+f(x+1_j)\ge f(x+1_i)+f(x+2_j) \tag{48}$$

holds for all x,i,j, $i\ne j$, such that all four vectors involved are in $B(a)$.

Here 2_j stands for $2\cdot 1_j$, and note that i,j need not be ordered. So the skew-submodularity imposes a restriction on f within each sub-box of the form $B(x|\,x+1_i+2_j)$ in $B(a)$. The picture illustrates the corresponding tiling of the zonogon $Z(1_i+1_j)$ when $i<j$ (on the right) and $j<i$ (on the left); here the skew-submodular condition reads as $f(B)+f(C)\ge f(A)+f(D)$.



In fact, one can regard (48) as a degenerate form of the TP3-relation (3). Indeed, putting $j=k$ in (3), we obtain

$$f(x+1_i+1_j)+f(x+1_j)=\max\{f(x+1_i+1_j)+f(x+1_j),\,f(x+2_j)+f(x+1_i)\},$$

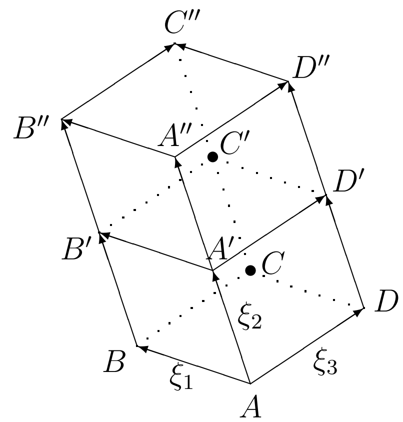
which is just equivalent to (48).

Theorem 7.1 *A TP-function f on a box $B(a)$ is skew-submodular if and only if its restriction to the standard basis $Int(a)$ is skew-submodular (in the sense that (48) holds whenever $i\ne j$ and the four vectors occurring in it belong to $Int(a)$). Furthermore, a skew-submodular f satisfies the additional relations:*

$$f(x+1_i+1_j)+f(x+1_j+1_k)\ge f(x+1_i+1_k)+f(x+2_j), \tag{49}$$

where i,j,k are different.

Proof. Arguing as in the previous section and using Propositions 5.1 and 5.3, we reduce the task to examination of the 3-dimensional boxes $B(1,1,2)$, $B(1,2,1)$ and $B(2,1,1)$. Below we consider the case $B(1,2,1)$ (in the other two cases, the proof is analogous and we leave it to the reader as an exercise). This case is illustrated in the picture:



There are two TP3-relations in the box, namely:

$$f(A') + f(C) = \max(f(B) + f(D'), f(B') + f(D)) \tag{50}$$

and

$$f(A'') + f(C') = \max(f(B') + f(D''), f(B'') + f(D')). \tag{51}$$

The face (parallelogram) $AA''B''B$ gives the skew-submodular inequality in the standard basis:

$$f(A'') + f(B) \le f(A') + f(B'). \tag{52}$$

The face $AA''D''D$ gives one more skew-submodular inequality

$$f(D) + f(A'') \le f(D') + f(A'). \tag{53}$$

First of all we prove inequality (49) (with $(i,j,k) = (1,2,3)$); it is viewed as

$$f(B') + f(D') \ge f(A'') + f(C). \tag{54}$$

Adding $f(D')$ to (52) gives

$$f(A'') + f(B) + f(D') \le f(A') + f(B') + f(D').$$

Adding $f(B')$ to (53) gives

$$f(D) + f(A'') + f(B') \le f(A') + f(D') + f(B').$$

These inequalities together with (50) result in

$$f(A'') + f(A') + f(C) \le f(A') + f(B') + f(D').$$

Now the desired inequality (54) is obtained by canceling $f(A')$ in both sides.

Next we show validity of the other two skew-submodular inequalities in the box, namely, those concerning the faces $BB''C''C$ and $DD''C''C$.

Adding (54) and the inequality $f(B'') + f(D') \le f(A'') + f(C')$ (which is a consequence of (51)), we obtain

$$f(A'') + f(C) + f(B'') + f(D') \le f(B') + f(D') + f(A'') + f(C').$$

Canceling $f(A'') + f(D')$ in this inequality gives

$$f(C) + f(B'') \le f(B') + f(C'),$$

which is just the skew-submodular inequality for the face $BB''C''C$. The skew-submodular inequality $f(C) + f(D'') \le f(D') + f(C')$ for the face $DD''C''C$ is obtained in a similar way. ■

8 Discrete concave TP-functions

In this section we combine the above submodular and skew-submodular conditions on TP-functions.

Let us say that a TP-function f on a box $B(a)$ is a *DCTP-function* if

$$f(x+1_i+1_j)+f(x+1_j+1_k)\ge f(x+2_j)+f(x+1_i+1_k) \tag{55}$$

holds for all $x\in B(a)$ and $i,j,k\in\{0\}\cup[n]$ such that the four vectors in this relation belong to $B(a)$. Here 1_0 means the zero vector. Note that i,j,k need not be ordered and some of them may coincide.

Remark 6. The meaning of the abbreviation "DC" is that the TP-functions obeying (55) possess the property of discrete concavity. More precisely, one can check that such functions satisfy requirements in a discrete concavity theorem from [15, Ch. 6], and therefore, they form a subclass of *polymatroidal concave functions*, or $M^\#$-concave functions, in terminology of that paper.

Observe that if $j=0\ne i,k$ and $i\ne k$, then (55) turns into the submodular condition (cf. (43)). If $k=0\ne i,j$ and $i\ne j$, then (55) turns into the skew-submodular condition (48). And if $i=k=0$, then (55) turns into the concavity inequality

$$2f(x+1_j)\ge f(x)+f(x+2_j).$$

One easily shows that this inequality follows from submodular and skew-submodular relations.

Now assume that none of i,j,k is 0. If all i,j,k are different, then (55) is a consequence of the skew-submodularity, due to Theorem 7.1. Finally, if $i=k$, then (55) turns into

$$2f(x+1_i+1_j)\ge f(x+2_i)+f(x+2_j),$$

which again is easily shown to follow from skew-submodular relations.

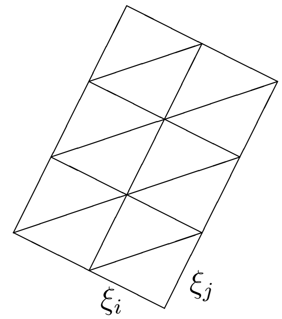
The above observations are summarized as follows.

Proposition 8.1 *A TP-function on a box is a DCTP-function if and only if it is submodular and skew-submodular.*

This proposition and Theorems 6.1 and 7.1 give the following

Corollary 8.2 *A TP-function f on a box $B(a)$ is a DCTP-function if and only if it is submodular and skew-submodular on the standard basis $Int(a)$.*

One can visualize this corollary by considering the standard tiling of the zonogon $Z(a)$ (i.e., the RT-diagram associated with the standard basis $Int(a)$). It contains "big" parallelograms $P(i,j)$ for $i<j$, where $P(i,j)$ is the sub-zonogon $Z(a_{i+1}\xi_{i+1}+\ldots+a_{j-1}\xi_{j-1}\ ;\ a_i1_i+a_j1_j)$. Subdivide each ij-rhombus $[x,x+\xi_i,x+\xi_j,x+\xi_i+\xi_j]$ in $P(i,j)$ into two triangles by drawing the diagonal $[x+\xi_i,x+\xi_j]$. This gives a triangulation of $P(i,j)$; see the picture where $a_i=2$ and $a_j=3$.



In terms of such triangulations, the submodular and skew-submodular conditions on f say that for any two adjacent triangles ABC and BCD, one holds $f(B) + f(C) \geq f(A) + f(D)$. In other words, the affine interpolation of f within each parallelogram $P(i,j)$ is concave.

Next we use the above description of DCTP-functions on $B(a)$ to obtain a characterization of DCTP-functions on a box truncated from above.

Proposition 8.3 *A TP-function f on a truncated box $B_0^{m'}(a)$ is a DCTP-function if and only if it is submodular and skew-submodular on the standard basis $\mathcal{B} = Int(a;1) \cup \ldots Int(a;m')$.*

Proof. For convenience we identify the elements of $B_0^{m'}(a)$ with their projections in the zonogon $Z(a)$. Then the basis $\mathcal{B}$ consists of the vertices x of the standard tiling of $Z(a)$ such that $|x| \leq m'$. The submodularity and skew-submodularity of f on $\mathcal{B}$ means that, in the triangulation of each parallelogram $P(i,j)$ as above, the inequality $f(B)+f(C) \geq f(A)+f(D)$ holds for every two adjacent triangles ABC and BCD of the triangulation, with all A,B,C,D occurring in $\mathcal{B}$. We assert that the restriction of f to $\mathcal{B}$ can be extended to a function $\widetilde{f}$ on the standard basis $Int(a)$ of the entire box $B(a)$ such that $\widetilde{f}$ is submodular and skew-submodular.

With respect to $\mathcal{B}$, there are three groups of "big" parallelograms of the standard tiling of $Z(a)$. The first group consists of those $P(i,j)$ which have all vertices in $\mathcal{B}$, the second one consists of the $P(i,j)$'s having vertices in $\mathcal{B}$ and not in $\mathcal{B}$, and the third one consists of the $P(i,j)$'s with all vertices not in $\mathcal{B}$.

We order the parallelograms of the second group clockwise; then two consecutive parallelograms share a common edge. The following claim will be of use.

Claim. *Let h be a discrete concave function on a 2-dimensional truncated box $B_0^c(0|\,(a,b))$. When $c < b$, let g be a concave function on the segment $[c,b]$ such that $h(0,c) = g(c)$ and $h(0,c-1) - h(0,c) \geq g(c) - g(c+1)$. Then there exists a discrete concave function $\widetilde{h} : B(0|\,(a,b)) \to \mathbb{R}$ such that $\widetilde{h}$ coincides with h on $B_0^c(0|(a,b))$ and $\widetilde{h}(0,k) = g(k)$ for all $k \in [c,b]$. Symmetrically, when $c > b$, let g be a concave function on the segment $[c-b,a]$ such that $h(c-b,b) = g(c-b)$ and $h(c-b-1,b) - h(c-b,b) \geq g(c-b) - g(c-b+1)$. Then there exists a discrete concave function $\widetilde{h} : B(0|\,(a,b)) \to \mathbb{R}$ such that $\widetilde{h}$ coincides with h on $B_0^c(0|\,(a,b))$ and $\widetilde{h}(k,b) = g(k)$ for all $k \in [c-b,a]$.*

Proof of the Claim. Consider the case $c < b$. W.l.o.g., we may assume that h is

monotone (otherwise we add to h an appropriate separable discrete concave function). Define $\widetilde{h}$ by

$$\widetilde{h} = h * \widetilde{g} * \phi\{x \geq 0\},$$

where: $*$ denotes the convolution of two functions (namely, $f_1 * f_2(z) = \max_{x+y=z}\{f_1(x) + f_2(y)\}$); $\phi\{x \geq 0\} := 0$ for $x \geq 0$ and $-\infty$ otherwise; and $\widetilde{g} = g * \phi\{y \geq 0\}$.

Then $\widetilde{h}$ is a discrete concave function (as the convolution of discrete concave functions is discrete concave as well; see, e.g., [15]). Obviously, $\widetilde{h}(0,k) = g(k)$. By the discrete concavity of h, we have $h(x, c-x-1) - h(x, c-x) \geq h(0, c-1) - h(0, c)$, and since $h(0, c-1) - h(0, c) \geq g(c) - g(c+1)$, we obtain $h(x, c-x-1) - h(x, c-x) \geq g(c) - g(c+1)$. This implies that $\widetilde{h}$ coincides with h on $B_0^c(0|\,(a,b))$. In case $c > b$, we argue in a similar way. ■

Using the Claim, we extend f, step-by-step in the clockwise order, into a function which is discrete concave on each parallelogram of the second group.

Now consider a wiring associated to the big parallelograms (cf. Section 5). Then the parallelograms of the third group induce a connected sub-wiring. The extension of f to the parallelograms of the second group assigns (1-dimensional) discrete concave boundary values to the parallelograms of the third group, at most one boundary function being assigned to each wire. Thus, we can extend these boundary values to a separable discrete concave function on each parallelogram of the third group.

This completes the proof of the proposition. ■■

As a consequence of this proposition, we obtain the following result; it was announced without proof in [11].

Corollary 8.4 *A TP-function f is discrete concave on a simplex $B_m^m(m^n)$ if and only if it is submodular and skew-submodular on the standard basis $\mathcal{B} = Sint(m^n; m) \cup Int(m^n; m)$.*

Proof. Consider the projection of $B_m^m(m^n)$ into $\mathbb{R}^{n-1}$ along the first coordinate. Then f becomes a TP-function on the truncated box $B_0^m(m^{n-1}))$, and $\mathcal{B}$ is projected to $Int(m^{n-1}; 1) \cup \ldots \cup Int(m^{n-1}; m)$. Now the result follows from Proposition 8.3. ■

9 The tropical Laurent phenomenon

A collection of functions on a set $\mathcal{X}$ is said to possess the *Laurentness property* w.r.t. a subset $\mathcal{B} \subset \mathcal{X}$ if the values of these functions on the elements in $\mathcal{X} - \mathcal{B}$ are expressed as Laurent polynomials (depending on elements but not on functions) in the values on $\mathcal{B}$, whereas the latter values are usually assumed to be "independent". For the Laurent phenomenon under the octahedron and cube recurrences, see [8, 11, 19].

Its tropical analogue, the *tropical Laurentness property*, means that the value of a function f on an element $x \in \mathcal{X} - \mathcal{B}$ is expressed as

$$f(x) = \max_{i=1,\ldots,N} \sum\nolimits_{y \in \mathcal{B}} h_{i,y} f(y),$$

where the coefficients $h_{i,y}$ are integers depending on x (but not on f). In other words, $f(x)$ is represented by a piece-wise linear convex function of which arguments are the values of f on the elements of $\mathcal{B}$.

In what follows we explain that the TP-functions possess the tropical Laurentness property w.r.t. the standard basis, and moreover, estimate the coefficients in the corresponding "tropical Laurent polynomials".

1. We start with the TP-functions on the cube $C = 2^{[n]}$. The fact that $\mathcal{T}(C)$ possesses the tropical Laurentness property w.r.t. the standard basis Int (consisting of the intervals in $[n]$) easily follows from results in Section 3.

Indeed, by Theorem 3.1 and Proposition 3.2, a function $f \in \mathcal{T}(C)$ one-to-one corresponds to an $n \times n$ weight matrix W as in (12) (in our case $m = 0$ and $W = W'$), and the value of f on a set $S \subseteq [n]$ is viewed as

$$f(S) = \max\{w(\mathcal{F}) \colon \mathcal{F} \in \Phi_S\}, \tag{56}$$

where Φ_S is the set of admissible flows for S (i.e., beginning at the source set $\{s_p \colon p \in S\}$); cf. (11). Notice that each flow $\mathcal{F}$ in this expression is essential. Indeed, if we put $w_{pq} := 1$ for all vertices v_{pq} with $p \geq q$ covered by $\mathcal{F}$, and $w_{pq} := 0$ for the other vertices in the grid $\Gamma_{n,n}$, then $\mathcal{F}$ is the unique maximum-weight flow in Φ_S for this matrix W. So the number of linear pieces (slopes) in (56) is just $|\Phi_S|$.

According to (17), the weight w_{pq} of each vertex v_{pq} can be expressed, by a linear form, via the values of f on intervals:

$$w_{pq} = \sum\nolimits_{I \in Int} h_{pq}(I) f(I), \tag{57}$$

where each coefficient $h_{pq}(I)$ is 0,1 or –1 (h_{pq} is zero when $p < q$). Taking the sum of weights w_{pq} over the set $\Pi(\mathcal{F})$ of pairs pq concerning $\mathcal{F}$ and substituting it into (56), we obtain the desired tropical Laurent polynomial:

$$f(S) = \max\left\{\sum\nolimits_{I \in Int} h_{\mathcal{F}}(I) f(I) \colon \mathcal{F} \in \Phi_S\right\}, \tag{58}$$

where $h_{\mathcal{F}} := \sum(h_{pq} \colon pq \in \Pi(\mathcal{F}))$.

We assert that all coefficients $h_{\mathcal{F}}(I)$ are between –1 and 2.

To show this, consider a path P in $\mathcal{F}$. For an intermediate vertex $v = v_{pq}$ of P, we say that P makes *right turn* at v if the edge e of P entering v is horizontal (i.e., $e = (v_{p+1,q}, v)$) while the edge e' leaving v is vertical (i.e., $e' = (v, v_{q+1})$), and say that P makes *left turn* at v if e is vertical while e' is horizontal. Also if the first edge of P is horizontal, we (conditionally) say that P makes left turn at its beginning vertex as well. Let h_P denote the sum of functions h_{pq} over the vertices v_{pq} contained in P. The values of h_P on the intervals can be calculated by considering relations in (17) and making corresponding cancelations when moving along the path P. More precisely, one can see that

(59) (i) if P makes left turn at v_{pq}, then $h_P([p-q+1..p]) = 1$ and $h_P([p-q+1..p-1]) = -1$ (unless $q = 1$, in which case the interval $[p-q+1..p-1]$

vanishes); (ii) if P makes right turn at v_{pq}, then $h_P([p-q+1..p])=-1$ and $h_P([p-q+1..p-1])=1$; and (iii) $h_P(I)=0$ for the remaining intervals I in $[n]$.

This enables us to estimate the values of $h_{\mathcal{F}}$, i.e., of the sum of the functions h_P over the paths P in $\mathcal{F}$. Consider an interval $I=[c..d]$. Since the paths in $\mathcal{F}$ are disjoint, (59) shows that there are at most two paths P such that $h_P(I)\ne 0$. Therefore, $|h_{\mathcal{F}}(I)|\le 2$. Suppose $h_{\mathcal{F}}(I)=-2$. Then $h_P(I)=h_{P'}(I)=-1$ for some (neighboring) paths P,P' in $\mathcal{F}$. In view of (59), this can happen only if one of these paths makes right turn at the vertex v_{pq} with $p=d$ and $q=d-c+1$, while the other path makes left turn at the vertex $v_{p+1,q+1}$. But then P,P' must intersect; a contradiction. See Fig. 4(a).

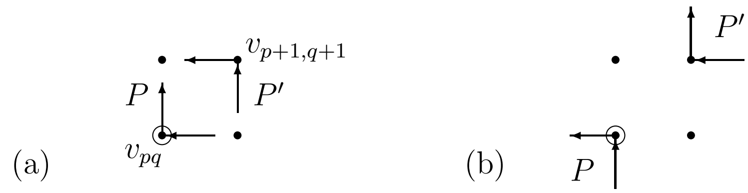


Figure 4: (a) $h_{\mathcal{F}}(I=[p-q+1..p])=-2$; (b) $h_{\mathcal{F}}([p-q+1..p])=2$.

Thus, $-1\le h_{\mathcal{F}}(I)\le 2$, as required. (In fact, $h_{\mathcal{F}}(I)=2$ is possible; in this case there are two paths in $\mathcal{F}$, one making left turn at $v_{d,d-c+1}$, and the other making right turn at $v_{d+1,d-c+2}$. See Fig. 4(b).) Summing up the above reasonings, we obtain the following

Proposition 9.1 *The set of TP-function on the cube $2^{[n]}$ possesses the tropical Laurentness property w.r.t. Int. This is expressed by (58) for each $S\in 2^{[n]}-Int$. Moreover, each coefficient $h_{\mathcal{F}}(I)$ in this expression is in $\{-1,0,1,2\}$.*

(Note that the lower and upper bounds –1 and 2 on the "tropical monomial" coefficients in this expression are similar to those on the exponents of face variables established by Speyer and stated in the Main Theorem of [19], where algebraic Laurent polynomials are considered.)

Remark 7. Adding an appropriate expression to each sum in the maximum, one can re-write (58) in the form

$$
\begin{aligned}
f(S)=\max\Big\{\sum\nolimits_{I\in Int} h'_{\mathcal{F}}(I)f(I)\colon \mathcal{F}\in\Phi_S\Big\}\\
-\sum(f(I)\colon I\in Int,\ I\subseteq[\min(S)+1..\max(S)-1]),
\end{aligned}
$$

where all coefficients $h'_{\mathcal{F}}(I)$ are nonnegative integers not exceeding 3. For example, for 2-element sets $ik=\{i,k\}$, $i<k$, one can obtain

$$
\begin{aligned}
f(ik)=\max\nolimits_{i<j<k}\{f(i)+\ldots+f(j-1)+f(j,j+1)+f(j+2)+\ldots+f(k)\}\\
-f(i+1)-\ldots-f(k-1).
\end{aligned}
$$

Remark 8. The admissible flows figured in (58) can be replaced by somewhat more transparent objects. Let us say that a triangular array $A = (a_{ij})$, $1 \le j \le i \le k$, of size k is a *semi-strict Gelfand-Tsetlin pattern* if $a_{i,j-1} < a_{ij} \le a_{i+1,j-1}$ holds for all i, j. (Classical Gelfand-Tsetlin patterns [9], or GT-patterns, are defined by the non-strict inequalities in both sides.) The tuple $a_{11} < a_{21} < \ldots < a_{k1}$ is called the *shape* of A. For each $S \subseteq [n]$, there is a bijection between the set Φ_S of admissible flows for S and the set of semi-strict GT-patterns of size $|S|$ with the shape $p_1 < \ldots < p_{|S|}$, where $S = \{p_1, \ldots, p_{|S|}\}$.

Indeed, given $\mathcal{F} \in \Phi_S$, let P_i be the path in $\mathcal{F}$ beginning at s_{p_i}. Let V_i be the set of vertices entered by vertical edges of P_i plus the source s_{p_i}. The second coordinate of the vertices in V_i runs from 1 through i (along P_i) and we denote these vertices as $v_{a_{ij},j}$, $j = 1, \ldots, i$. Then the admissibility of $\mathcal{F}$ implies that the arising triangular array (a_{ij}) (of size $|S|$) is a semi-strict GT-pattern. Conversely, given a semi-strict pattern A of size k with $a_{k1} \le n$, one can uniquely construct an admissible flow $\mathcal{F}$ in which the vertices entered by vertical edges are just $v_{a_{ij},j}$ for $i = 1, \ldots, k$ and $j = 2, \ldots, i$, and the sources are $v_{a_{i1},1} = s_{a_{i1}}$, $i = 1, \ldots, |S|$.

For a semi-strict GT-pattern A of shape $p_1 < \ldots < p_k \le n$ and a TP-function f on $2^{[n]}$, define

$$\widehat{f}(A) := \sum\nolimits_{i,j} \Delta f([a_{ij} - j + 1..a_{ij}]),$$

where for an interval $I = [c..d]$,

$$\Delta f(I) := f(I) + f(I - \{c, d\}) - f(I - \{c\}) - f(I - \{d\})$$

if $c < d$, and $\Delta f(I) := f(I)$ if $c = d$ (assuming $f(\emptyset) = 0$). One can check that for an admissible flow $\mathcal{F}$ and its corresponding semi-strict GT-pattern A, $\widehat{f}(A)$ is equivalent to $\sum_{I \in Int} h_{\mathcal{F}}(I)$. This and Proposition 9.1 give the following

Corollary 9.2 *For a TP-function f on $2^{[n]}$ and a subset $S = \{p_1, \ldots, p_{|S|}\} \subseteq [n]$ with $p_1 < \ldots < p_{|S|}$, one holds*

$$f(S) = \max\ \widehat{f}(A),$$

where the maximum is taken over all semi-strict GT-patterns A with the shape $p_1 < \ldots < p_{|S|}$.

2. Next we consider a truncated cube $C_m^{m'} \subset 2^{[n]}$. In this case the tropical Laurentness property for the TP-functions w.r.t. the standard basis $\mathcal{B}$ is shown in a similar way as for the entire cube $2^{[n]}$. There are only two differences. The first one is that expression (57) should be modified if $q \le m$ (and $p > m$). Now the weight w_{pq} is expressed by involving corresponding sesquialteral intervals according to the map π in (17). The second difference is that for each set $S \in C_m^{m'} - \mathcal{B}$, one should consider only those admissible flows $\mathcal{F}$ for S that cover all vertices v_{pq} with $p, q \le m$ (for otherwise $w(\mathcal{F})$ tends to $-\infty$ when the number M in (13) increases). We call such a flow *strong* and denote the set of strong flows for S by $\Phi_S^{\rm st}$. Then the statement in Proposition 9.1 is modified as follows.

Proposition 9.3 *For the TP-functions f on $C_m^{m'}$,*

$$f(S) = \max\left\{\sum\nolimits_{X\in\mathcal{B}} h_{\mathcal{F}}(X)f(X) \colon \mathcal{F}\in\Phi_S^{\rm st}\right\}$$

holds for each set $S\in C_m^{m'}$, where $\mathcal{B}$ is the standard basis for $C_m^{m'}$. Also each coefficient $h_{\mathcal{F}}(X)$ in this expression is in $\{-1,0,1,2\}$.

3. Finally, we consider an n-dimensional box $B(a)$. To show the Laurentness property for $\mathcal{T}(B(a))$ w.r.t. the standard basis $Int(a)$ (consisting of the fuzzy-intervals), we follow the method in Section 4, by embedding $B(a)$ into the Boolean cube $C=2^{[N]}$, where $N=|a|$, and then use the Laurentness property for the latter.

For a vector $x\in B(a)$, define the subset $[x]\subseteq[N]$ by (24). By Proposition 4.1, for each TP-function f on $B(a)$, there exists a TP-function g on C such that

$$f(x)=g([x]) \qquad \text{for all } x\in B(a).$$

Recall that the function g constructed in the proof of Proposition 4.1 is such that

$$g(I)=f(\#(I))+M\epsilon(I) \qquad \text{for each } I\in Int_N,$$

where $\epsilon(I)$ is the excess of I, and M a large positive integer (cf. (28)). By Proposition 9.1, $g([x])$ is expressed via a tropical Laurent polynomial in variables $g(I)$, where I runs over the set Int_N of intervals in $[N]$. Also the vector $\#(I)$ is a fuzzi-interval for each $I\in Int_N$. Since the convexity preserves under affine transformations of variables, we obtain that for each vector $x\in B(a)-Int(a)$, the values $f(x)$ for $f\in\mathcal{T}(B(a))$ are represented by a piece-wise linear convex function of the arguments $f(y)$, $y\in Int(a)$, and therefore, $\mathcal{T}(B(a))$ has the Laurentness property w.r.t. $Int(a)$.

More precisely, $f(x)$ is expressed by the tropical Laurent polynomial

$$f(x)=\max_{\mathcal{F}\in\Phi_{[x]}}\sum\nolimits_{I\in Int_N} h_{\mathcal{F}}(I)(f(\#(I))+M\epsilon(I)), \tag{60}$$

where, as before, Φ_S denotes the set of flows in C with the source set corresponding to S.

It remains to estimate the coefficients in the tropical monomials in (60). The sum concerning a flow $\mathcal{F}\in\Phi_{[x]}$ is of the form $\sum(\alpha_{\mathcal{F}}(y)\colon y\in Int(a))+\beta_{\mathcal{F}}M$ for some integers $\alpha_{\mathcal{F}}(y),\beta_{\mathcal{F}}$. Since M is large, $\beta_{\mathcal{F}}\le 0$. Also if $\beta_{\mathcal{F}}<0$, then the flow $\mathcal{F}$ can be ignored. So we can consider in (60) only the flows $\mathcal{F}$ with $\beta_{\mathcal{F}}=0$; we call them *regular* and denote the set of these by $\Phi_{[x]}^{\rm ref}$.

For $i=1,\ldots,n$, let Π_i denote the set of pairs pq such that $\overline{a}_{i-1}+1<p\le\overline{a}_i$ and $q<p-\overline{a}_{i-1}$, and let Γ_i be the triangular sub-grid of $\Gamma_{N,N}$ induced by the vertices v_{pq} for $pq\in\Pi_i$. Consider a regular flow $\mathcal{F}$ and a path $P\in\mathcal{F}$; let $h_P(I)$ ($I\in Int_N$), $\alpha_P(y)$ ($y\in Int(a)$) and β_P be the corresponding numbers for P. When P makes a turn at a vertex $v=v_{pq}$, we denote the interval $[p-q+1..p]$ by I_{pq}, and $[p-q+1..p-1]$ by I'_{pq}; then the contribution to β_P from v is $\beta_{pq}:=\epsilon(I'_{pq})-\epsilon(I_{pq})\le 0$ in case of right turn, and $\beta_{pq}:=\epsilon(I_{pq})-\epsilon(I'_{pq})\ge 0$ in case of left turn.

Claim. *(i) P cannot make any turn within $\Gamma_1 \cup \ldots \cup \Gamma_n$, and (ii) $\beta_P = 0$.*

Proof. Suppose P makes a turn at a vertex of some Γ_i. Let P begin at a block L_j (more precisely, at a source s_r with $r \in L_j$). The case $i = j$ is impossible (since $[x] \cap L_i$ consists of the first x_i elements of L_i, whence the paths beginning at L_i cannot make any turn before the "diagonal" $\{v_{p'q'} \colon p' \in L_i,\, q' = p' - \overline{a}_{i-1}\}$). Therefore, $j > i$. Let P make turns at the sequence $v_{p(1)q(1)}, \ldots, v_{p(k)q(k)}$ of vertices in Γ_i; then $p(1) - q(1) > \ldots > p(k) - q(k) > 0$. Clearly the edge entering $v_{p(1)q(1)}$ is horizontal; so P makes right turn at $v_{p(1)q(1)}$. For $d = 1, \ldots, k$, we have $\beta_{p(d)q(d)} = -(p(d) - q(d))$ if d is odd, and $\beta_{p(d)q(d)} = p(d) - q(d)$ if d is even (taking into account that P makes right turn at $v_{p(1)q(1)}$, that the size of the head of $I_{p(d)q(d)}$ is greater by 1 than that of $I'_{p(d)q(d)}$, and the shift of the head of each of these intervals is equal $p(d) - q(d)$). This implies that the contribution to β_P from the vertices of P within Γ_i is strictly negative.

Also if P makes a turn at an intermediate vertex v_{pq} not in $\Gamma_1 \cup \ldots \cup \Gamma_n$, then $\beta_{pq} = 0$ (since one can see that either I_{pq}, I'_{pq} have equal heads, or the shifts of the heads of these intervals are zero, or both, yielding equal excesses). And if P makes left turn at its beginning vertex, v_{p1} say, then p is the beginning of a block, and therefore, $\beta_{p1} = 0$. Thus, $\beta_P \le 0$. Now the claim follows from the fact that $\sum(\beta_P \colon P \in \mathcal{F})$ amounts to $\beta_{\mathcal{F}} = 0$. ∎

From (i) in the Claim it follows that for each fint $y \in Int(a)$ with $|\mathrm{supp}(y)| = 1$, if $\alpha_{\mathcal{F}}(y) \ne 0$, then there is exactly one interval $I \in Int_N$ such that $\#(I) = y$ and $h_{\mathcal{F}}(I) \ne 0$. A similar property is trivial if $|\mathrm{supp}(y)| > 1$. This implies $\alpha_{\mathcal{F}}(y) = h_{\mathcal{F}}(I)$, whence $-1 \le \alpha_{\mathcal{F}}(y) \le 2$. Summing up the above reasonings, we conclude with the following

Proposition 9.4 *The set of TP-function f on a box $B(a)$ possesses the tropical Laurentness property w.r.t. the set $Int(a)$ of fuzzy-intervals. For each $x \in B(a) - Int(a)$, the value $f(x)$ is expressed as*

$$f(x) = \max\nolimits_{\mathcal{F} \in \Phi^{\mathrm{reg}}_{[x]}} \sum\nolimits_{I \in Int_N} h_{\mathcal{F}}(I) f(\#(I)),$$

and all coefficients in the tropical monomials of this expression are in $\{-1, 0, 1, 2\}$.

References

[1] A. Berenstein, S. Fomin, and A. Zelevinsky, Parametrizations of canonical bases and totally positive matrices, *Adv. Math.* **122** (1996), 49-149.

[2] N.G. de Bruijn, Dualization of multigrids, *J. Phys. France* **47** (1986) 3–9.

[3] V. Danilov and G. Koshevoy, The octahedron recurrence and RSK-correspondence, *Séminaire Lotharingien de Combinatoire* **54A**, Article B54An, 2007.

[4] A.M. Dress and W. Wenzel, Valuated matroids, *Adv. Math.* **91** (1992) 158208.

[5] P.H. Edelman and V. Reiner, Free arrangements and rhombic tilings, *Discrete Comput. Geometry* **15** (1996) 307–340.

[6] V. Fock, A. Goncharov, Moduli space of local systems and higher Teichmmuller theory, *Publ. Math. IHES*, **103** (2006) 1–211; *ArXiv*:math.AG/9909.077.

[7] W. Fulton, *Young tableaux*, London Math. Soc. Student Text, Vol. **35**, Cambridge Univ. Press, Cambridge, 1997.

[8] S. Fomin, and A. Zelevinsky, The Laurent phenomenon, *Adv. Applied Math.* **28** (2) (2002), 119–144.

[9] I.M. Gel'fand and M.L. Tsetlin, Finite-dimensional representations of the group of unimodular matrices, *Doklady Akad. Nauk SSSR* **71** (1950), 825–828, in Russian.

[10] A. Henriques, An action of the cactus group, *ArXiv*:math.AG/0705.3000, 2007.

[11] A. Henriques and D.E. Speyer, The multidimensional cube recurrence, *ArXiv*:0708.2478v1[math.CO], 2007.

[12] J. Kamnitzer, The crystal structure on the set of Mirkovic-Vilonen polytopes, *ArXiv*:math.QA/0505.398, 2005.

[13] A. Knutson, T. Tao, and C. Woodward. A positive proof of the Littlewood-Richardson rule using the octahedron recurrence, *Electron. J. Combin.*, v. **11**, Research Paper 61, 2004; *ArXiv*:math.CO/0306274, 2003.

[14] L. Lafforgue, Chirurgie des grassmanniennes, *CRM Monograph Series*, v. **19**, AMS, 2003.

[15] K. Murota, *Discrete Convex Analysis*, SIAM, Philadelphia, 2003.

[16] R.C. Penner and G.Mcshane, Stable curves and screens on fatgraphs, *ArXiv*:0707.1468 [math.GT], 2007.

[17] A. Postnikov, Total positivity, Grassmannians, and networks, *ArXiv*:math.CO/0609764, 2006.

[18] J. Scott, Grassmannians and cluster algebras, *ArXiv*:math.CO/0311148, 2003.

[19] D. Speyer, Perfect matchings and the octahedron recurrence, *Journal of Algebraic Combinatorics*, 25 (2007), 309-348.

[20] D. Speyer, L. Williams, The tropical totally positive Grassmanian, *ArXiv*:math.CO/0312.297, 2003.